The Power of Quantitative Easing: Liquidity versus Interest Rate Channel

Xiang SHI*

Department of Economics, Hong Kong University of Science and Technology

May 20, 2024

The newest version of this paper

Abstract

This paper presents a novel decomposition of the transmission of quantitative easing (QE) into two distinct channels: the *Liquidity* channel, which involves injecting liquidity into the market, and the *Interest Rate* channel, which concerns the manipulation of the term structure of interest rates. Using a general equilibrium model that includes household heterogeneity and financial frictions, I show how these elements affect the effectiveness of QE by asymmetrically altering the strength of these two channels. After quantitatively solving and calibrating the model, my findings indicate that the liquidity channel's contribution to QE's stimulative impact on output is approximately 1.5 times greater than that of the interest rate channel. Additionally, the interplay between household heterogeneity and financial friction is crucial in determining the effect of QE, rather than one factor alone. Finally, I empirically validate these channels by introducing a novel instrumental variable and an identification methodology within a Bayesian-IV-VAR framework. The empirical evidence aligns with the quantitative findings, revealing that the liquidity channel is 1.46 times more effective than the interest rate channel.

JEL classification: C11, E21, E30, E52, E58

Keywords: HANK, Monetary Policy, Quantitative Easing, Financial Friction, Bayesian VAR, IV-VAR

*PhD student in Hong Kong University of Science and Technology. Email: xiang@xiangshi.email. I am deeply indebted to Byoungchan Lee, Marc Dordal Carreras, and Jenny Xu for their invaluable support and generous altruistic help. I am also grateful to ChaeWon Baek, Wei Cui, Jun Yu, Juyoung Yang , and other audiences in seminars and conferences for their priceless comments and suggestions. Any remaining errors or omissions are my own.

1 Introduction

Because of the ZLB, the conventional monetary policy fails to stimulate the economy from the recession. Therefore, during the ZLB period, the Federal Reserve implemented unconventional monetary policy by purchasing long-term Treasury bonds and mortgage-backed securities from the market, which is also referred to as a large asset purchasing policy. Asset purchasing, or quantitative easing, twists the term yield and drives up the price of long-term bonds by increasing the demand for Treasury bonds. There are several mechanisms[1] through which QE stimulates the economy, and in this paper, I argue that the effect of QE on macroeconomics can be divided into two main channels: the interest rate channel and the liquidity channel. Other mechanisms will influence the effect of QE by symmetrically or asymmetrically influencing these two channels. As argued by the Lucas Critique, due to the endogeneity between monetary policy and macroeconomics, understanding monetary policy is crucial for enhancing its effectiveness, facilitating economic estimation and prediction, and stabilizing the economy. By decomposing and analyzing these two channels, we can enhance our understanding and discern the working principles of QE more clearly.

These two channels are related to the two outcomes of the central bank's asset purchasing process: increased holding of long-term bonds (of central bank) and decreased long-term yields. The central bank achieves the former by injecting liquidity, cash, into the financial market, and attains the latter one by increasing the demand for long-term bonds. Correspondingly, I outline liquidity injection as the *liquidity channel*, and term-yield twisting as the *interest rate channel*[2]. The output is stimulated by the liquidity channel as financial institutions in the financial market obtain more cash from selling bonds and alleviate underinvestment or capital misallocation caused by financial friction, resulting in more investment and increased output. Conversely, the interest rate channel stimulates output as the drop in long-term rates changes people's expectations about future interest rates and the returns on long-term bonds, leading financial institutions to invest more in capital or equity markets due to the non-arbitrage condition, which generates portfolio adjustment and additional output. In addition to these direct effects, there are some indirect effects through the general equilibrium as they also operate in conventional monetary policy. An income jump from increased investment and higher labor demand relaxes the constraints of hand-to-mouth households, who then increase their consumption due to higher disposable income, further stimulating output.

To understand the channels through which unconventional monetary policy operates, I first

[1] Kuttner (2018) summarized them in detail. Imperfect sustainability states that market segmentation exists, such that long-term bonds with different maturities cannot be freely substituted for each other. The signaling channel suggests that changes in the expectation of future short-term interest rates will affect the price of long-term bonds. The financial balance sheet channel implies that changes in the balance sheets of financial institutions will affect their demand for long-term bonds of different maturities, thereby altering the prices of these bonds.

[2] If we take the the effect of QE as function $Y(BQ)$ where Y is the output, B is the volume of long-term bonds and Q is the price of the bonds. Then by total differential $\frac{\partial Y}{\partial BQ} = \frac{\partial Y}{\partial BQ}\overline{Q} + \frac{\partial Y}{\partial Q}\overline{BQ}$ where $\frac{\partial Y}{\partial BQ}$ is the liquidity channel and $\frac{\partial Y}{\partial Q}$ is the interest rate channel.

employ a general equilibrium model with heterogeneous households and a financial accelerator to investigates how unconventional monetary policy stimulates the economy through these two channels. Apart from the wealthy households whose collateral is unbounded, two types of hand-to-mouth households—poor and wealthy—also exist within the economy, though their consumption is subject to their income each period. Only the change in wage income will influence the consumption of the poor hand-to-mouth households, yet changes in both wage and asset return will influence the consumption of wealthy hand-to-mouth households. Similar to the arguments for conventional monetary policy by Kaplan et al. (2018) and Auclert (2019), the presence of heterogeneous households and hand-to-mouth agents amplifies the impact of unconventional monetary policy through general equilibrium and indirect effects. Even at the zero lower bound (ZLB) where the interest rate is fixed and household consumption cannot be stimulated through the conventional monetary policy, hand-to-mouth households, with a high marginal propensity to consume (MPC), increase consumption and stimulate the economy, as the standard Euler equation does not apply to them and their consumption is governed by the budget constraint. Moreover, the change in the return of long-term bonds and equity will be passed to the change in illiquid asset return as the wealthy households invest in the financial market via financial institutions. Hence, the consumption of wealthy hand-to-mouth households and the output is further stimulated through the change in illiquid asset return. Additionally, there is a novel redistribution mechanism in this paper, created by the heterogeneous households in the liquidity channel, as the liquidity used by the central bank is sourced from wealthy non-hand-to-mouth households, yet the benefit of a stimulated economy is enjoyed by the whole economy, both hand-to-mouth and non-hand-to-mouth households[3].

In addition to the amplification effect created by heterogeneous households on the demand side, the financial accelerator also expands the effectiveness of unconventional monetary policy by affecting the two channels. I introduce market segmentation (across different types of bonds and equity) through the workhorse model Gertler and Kiyotaki (2015) to investigate it in detail. In this model, financial institutions cannot hold arbitrary positions (both long and short) in short-term bonds, long-term bonds, or equity, but are subject to restrictions in LTV ratio. Due to these financial constraints, financial institutions cannot borrow as much as they would like to fulfill their ideal portfolio arrangement of assets. Therefore, under the liquidity channel, the financial institutions will use the injected liquidity from the central bank to buy more equity to stimulate output, rather than saving it or paying off debt. Meanwhile, because of the financial constraint and leverage ratio, they will invest more than one unit of money in capital with loans when they receive one unit of money from selling treasury bonds. Conversely, under the interest rate channel, the financial institutions can hardly invest in more equity when the return on long-term bonds drops, as they are financially constrained and lack the cash to buy new equity.

[3] Di Maggio et al. (2020)empirically tested the related effect and drew a conclusion that QE1 substantially increased refinancing activity and stimulated consumption, although there is no theoretical investigation of this effect.

However, they can still gain some extra liquidity to invest in equity (and thus stimulate the economy) through the interest rate channel by a pecuniary effect, because they previously held some amount of long-term bonds which have now become more valuable[4].

Household heterogeneity (financial accelerator) can augment the power of unconventional monetary policy through the two channels indirectly (directly) by general equilibrium (partial equilibrium in production sector). Moreover, together they generate a complementary effect that amplifies the power of unconventional monetary policy through both channels[5], as it is the wealthy households (savers) who own the financial institutions. The labor market boom not only increases the income and consumption by hand-to-mouth households but also increases the income of savers, who will invest in equity through the financial institutions, which further spirals up the economy and is amplified by the financial accelerator with more debt over leverage ratio. Thus, the complementarity between household heterogeneity and the financial accelerator is effective.

To quantify the relative effects of the liquidity and interest rate channels, I carefully calibrate the model to ensure it aligns with empirical evidence. After adjusting the parameters to turn off interest rate channel and extracting the pure effect of the liquidity channel, I conclude that quantitatively the effectiveness of the liquidity channel in stimulating output is significantly greater than that of the interest rate channel, with the former being approximately one and a half times larger than the latter. Additionally, by comparing different models with and without these elements, I demonstrate that neither household heterogeneity nor the financial accelerator alone significantly amplifies the power of unconventional monetary policy, but together they contribute to its amplification.

Additionally, I utilize a VAR model to decompose the effects of unconventional monetary policy into liquidity and interest rate channels by proposing a novel Bayesian IV-VAR method to identify the two channels of unconventional monetary policy. Previous literature has focused on the restrictions imposed on impulse responses and the second-stage regression of IV-VAR. However, in this paper, I introduce new instrument weakness restrictions on instrumental variables (first-stage regression) to further utilize the instruments. The weakness restrictions require that one instrument explains (varies) the corresponding structural shock more than it explains other shocks. These restrictions conserve degrees of freedom as they control the explanatory power of instruments, which is already considered during the instrument selection process. Therefore, we do not need to impose any additional restrictions in the main identification step (second-stage regression), which previously relied on economic intuition. Additionally, I introduce a new high-frequency instrumental variable, the treasury announcement change, to separately identify the liquidity and interest rate channels alongside FOMC changes. The empirical results support my quantitative findings and demonstrate that the ratio of the effectiveness of unconventional

[4]In this sense it is just the opposite effect of the channel that results in the bankrupt of Silicon Valley Bank in 2023.

[5]This is akin to the discussion under conventional monetary policy by Bilbiie et al. (2022).

monetary policy through the liquidity and interest rate channels is 1.46, closely aligning with my quantitative estimate of 1.5.

This paper contributes to the literature on Heterogeneous Agent New Keynesian (HANK) models and monetary policy, referencing seminal works such as McKay et al. (2016); Gornemann et al. (2016); Guerrieri and Lorenzoni (2017); Kaplan et al. (2018); Auclert (2019); Bayer et al. (2019); Hagedorn et al. (2019); Bilbiie (2020); Chang et al. (2021); Luetticke (2021). While most of these studies focus on investigating conventional monetary policy or solving the forward guidance puzzle, this paper centers on the indirect effects of quantitative easing that are amplified by household heterogeneity. In this way, my work modifies and complements that of Cui and Sterk (2021); Sims et al. (2022). Cui and Sterk (2021) analyzed the effect of QE in a heterogeneous-household model by focusing on the distributional MPC. While I contribute further to specifically stressing the importance of heterogeneity and . Additionally, they do not incorporate financial frictions in the financial sector into their HANK model and Sims et al. (2022) examined the contribution of heterogeneous households to the effectiveness of unconventional monetary policy. My work complements theirs by providing a tractable analytical analysis and focusing more on distinguishing the liquidity and interest rate channels, rather than the cross-effects of quantitative easing and household heterogeneity.

Furthermore, this paper also contributes to the literature related to unconventional monetary policy, referencing works such as Gertler and Karadi (2011); Krishnamurthy and Vissing-Jorgensen (2011); Carlstrom et al. (2017); Harrison (2017); Sims and Wu (2021). However, most of these studies have concentrated on the interaction between financial markets and monetary policy or on the optimal implementation of unconventional monetary policy. My paper extends this literature by exploring the cross-effects of financial friction and heterogeneous households on unconventional monetary policy, and investigates these through a novel decomposition.

This paper makes two empirical contributions, and firstly I improve the methodology related to IV-VAR. Stock and Watson (2012) and Mertens and Ravn (2013) initially proposed a new SVAR method incorporating instrumental variables, which was later expanded by Arias et al. (2021) and Giacomini et al. (2021) to combine the Bayesian method in IV-VAR. However, all of them adopted conventional restrictions on the contemporaneous response matrix of the shocks. I enhance this by fully utilizing the information provided by instrumental variables through a novel instrument weakness restriction, thereby identifying the shocks more precisely without any restrictions. Secondly, this paper proposes new instrumental variables, specifically the treasury announcement shock, to separately identify the liquidity and interest rate channels of unconventional monetary policy. Similarly to the standard FOMC announcement shock, I use the change in the price of future contracts within treasury bond issuance announcement windows as the instrument, known as the treasury announcement shock. During the announcement, the only information that the market receives is related to the liquidity of the treasury securities, which helps isolate the liquidity channel.

The remainder of this paper is organized as follows. Section 2 explains the baseline model in detail. Section 3 analytically discusses the liquidity and interest rate channel. After the theoretical analysis, in section 4 I provide the quantitative result related to the decomposition of these two channels. Then section 5 introduces the empirical support with the identification methodology and the description of instrument variables. In the end section 6 concludes the result.

2 Baseline Model

This section explains the baseline model in detail and illustrates how this general equilibrium model, featuring heterogeneous households and financial frictions, can reveal the operation of QE, through which the central bank purchases treasury bonds from the financial market to stimulate the economy. Based on this baseline model, I provide theoretical and quantitative results in the next section on how QE expands output through two different channels, each governed by distinct mechanisms.

2.1 Household

There are three types of household in the economy and i denotes their type following $i \in \{\mathrm{pHtM}, \mathrm{wHtM}, \mathrm{nHtM}\}$where pHtM states poor hand-to-mouth household.wHtM states wealthy hand-to-mouth household and respectively nHtM states non-hand-to-mouth household. Household i supply labour l_t^i to intermediate goods producers and earn real wage w_t with idiosyncratic shock ε_t^i at period t[6]. The government taxes τ portion of the total wage income to finance social welfare spending. Households can also hold two types of bond: short-term bonds b and illiquid asset a. The liquid bonds b_t brings a gross real return rate R_t realized at time $t+1$; the illiquid asset a_t brings a gross real return R_{t+1}^a realized at time $t+1$. Households can freely adjust their holding of liquid bonds b without transaction restriction but they cannot adjust the illiquid asset freely[7]. In addition to the interest rate and labour income households get lump sum tax transfer T_t and public unemployment insurance subsidy Θ_t.

There are three types of households in the economy, denoted by i where $i \in \{\mathrm{pHtM}, \mathrm{wHtM}, \mathrm{nHtM}\}$. Here, pHtMrepresents poor hand-to-mouth households, wHtM represents wealthy hand-to-mouth households, and \ nHtM represents non-hand-to-mouth households. Each household type i supplies labor l_t^i to intermediate goods producers and earns a real wage w_t with an idiosyncratic shock ε_t^i at period t[8]. The government taxes a τ portion of the total wage income to finance social welfare spending. Households can also hold two types of bonds: short-term bonds b and illiquid assets a. The liquid bonds b_t yield a gross real return rate R_t, realized at time $t+1$; the

[6]For simplicity I omit the agent index of household i henceforth since their optimization problem is isomorphic.

[7]As shown by Cui and Sterk (2021) the household selecting the illiquid asset withdrawing is equivalent to selecting the optimal illiquid asset level. Both of them can pin down the illiquid asset distribution path as long as they have the same starting point.

[8]For simplicity I omit the agent index of household i henceforth since their optimization problem is isomorphic.

illiquid asset a_t offers a gross real return R^a_{t+1}, also realized at time $t+1$. Households can freely adjust their holding of liquid bonds b without transaction restrictions but cannot freely adjust their holdings of illiquid assets[9]. In addition to the interest rate and labor income, households receive lump-sum tax transfers T_t and public unemployment insurance subsidies Θ_t.

Households at period t maximize their future discounted utility

$$
\begin{aligned}
V(b^i_{t-1}, a^i_{t-1}, \varepsilon^i) = & \max_{c_t, b_t, X^i_t} U(c^i_t, l^i_t) + \beta \mathbb{E} V(b^i_t, a^i_t, \varepsilon^i) \\
& \text{s.t. } c^i_t + b^i_t = X^i_t + b^i_{t-1} R_{t-1} + (1-\tau_l) w_t l_t \varepsilon^i_t + \Theta^i_t 1_{\varepsilon^i_t = 0} + T_t \\
& a^i_t \geq 0 \\
& R^a_t a^i_{t-1} - X^i_t = a^i_t
\end{aligned}
$$

The utility function is represented by the standard CRRA form $U(c, l) = \frac{c^{1-\sigma}}{1-\sigma} - \kappa \frac{l^{1+\psi}}{1+\psi}$, where σ is the inverse intertemporal elasticity of substitution, and ψ is the inverse Frisch elasticity of labor supply. κ measures the extent of disutility that labor generates.

X^i_t is the real illiquid asset extraction, which is exogenous and fixed in the baseline model[10]. Furthermore, I assume there are only two types of realization for the labor productivity shock ε^i_t for tractability.

$$
\varepsilon^i_t = \begin{cases} 0 & i \in \{\text{pHtM}, \text{wHtM}\} \\ 1 & i \in \{\text{nHtM}\} \end{cases}
$$

Because hand-to-mouth households do not have wage income, they cannot borrow as much as they would like to satisfy their optimal consumption decision, which is typically governed by the Euler equation. Therefore, their consumption level is solely determined by their budget constraint, and any change in the real interest rate cannot stimulate their consumption as they are financially constrained. Hence, the consumption of pHtM households is determined by lump-sum tax transfers and unemployment insurance, such that $c^{pHtM}_t = \Theta^{\text{HtM}}_t + T_t$. Similarly, the consumption of wHtM households is determined by $c^{\text{wHtM}}_t = X^{\text{wHtM}} + \Theta^{\text{HtM}}_t + T_t$. Compared to poor hand-to-mouth households, wealthy hand-to-mouth households have an additional income source from the illiquid assets they hold, which is X^{wHtM} in the baseline model. For non-hand-to-mouth households, their consumption is governed by the standard Euler equation, except for the precautionary saving term on the right-hand side, as now their future income and consumption

[9]As shown by Cui and Sterk (2021) the household selecting the illiquid asset withdrawing is equivalent to selecting the optimal illiquid asset level. Both of them can pin down the illiquid asset distribution path as long as they have the same starting point.

[10]As Cui and Sterk (2021) proved it is equivalent to pin down illiquid asset a^i_t or pin down extraction X^i_t as long as the economy has the same initial illiquid asset distribution. I use extraction instead of the illiquid asset in model henceforth because it is more tractable and helpful to simplify the model.

are uninsured.

$$U_c\left(c_t^{\text{nHtM}}\right)=\mathbb{E}\beta R_t\left\{p^{\text{nHtM}}U_c\left(c_{t+1}^{\text{nHtM}}\right)+p^{\text{pHtM}}U_c\left(c_{t+1}^{\text{pHtM}}\right)+p^{\text{wHtM}}U_c\left(c_{t+1}^{\text{wHtM}}\right)\right\} \tag{1}$$

Since only the wealthy hand-to-mouth households and non-hand-to-mouth households hold illiquid assets, these two types of agents can access the financial market to withdraw assets. This transaction closes the illiquid asset market by

$$X_t=h^{\text{nHtM}}X_t^{\text{nHtM}}+h^{\text{wHtM}}X_t^{\text{wHtM}}$$

where X_t represents the aggregate illiquid asset withdrawal.

2.2 Mutual funds

A continuum of surviving or newly entered mutual funds, indexed from 0 to 1, selects the share of firm equity $s_{j,t}$ and the amount of long-term real treasury bonds held $b_{j,t}^m$ at the end of period t, which will yield returns at time $t+1$[11]. At the beginning of period $t+1$, the aggregate shock is first realized before the production process occurs, when R_{t+1}^k and R_{t+1}^B are realized. By choosing the optimal portfolio arrangement, the mutual fund solves the problem

$$W\left(n_t|s_t^*,b_{,t}^{m*}\right)=\max_{s_{j,t},b_{j,t}^m}V\left(s_t,b_t^m,n_t\right) \tag{2}$$

$$\text{s.t. } V\left(s_t,b_t^m,n_t\right)\geq\lambda^vQ_ts_t+\lambda^b\lambda^vq_t^Bb_t^m \tag{3}$$

where the asterisk represents the variable evaluated at the optimal equilibrium level. W_t is the value of the surviving mutual fund, and V_t is the value of the mutual fund at the end of period t. λ^v and λ^b are parameters that regulate the collateral constraint, which implies that the market value of mutual fund j itself should not be lower than the assets they hold. λ^ν represents the strength of the collateral constraint, and λ^b denotes the relative strength between equity and treasury bonds that contributes to the collateral constraint.

After paying their borrowing costs, mutual funds will survive into the next period with a probability of θ^m and exit the financial market with a probability of $1-\theta^m$. Given the random exit-and-entry risk associated with mutual funds, W_t can be taken as the ex-post value of a mutual fund, and V_t as the ex-ante value, which is composed of two components: the return from those that exit and the expected future value of those that survive, such that

$$V\left(s_t,b_t^m,n_t\right)=E_t\beta\Lambda_{t,t+1}\left[\left(1-\theta^m\right)n_{t+1}+\theta^mW\left(n_{t+1}|s_{t+1}^*,b_{t+1}^{m*}\right)\right]$$

[11]For simplicity I omit the subscript j until aggregation.

The balance sheet of the mutual fund is

$$Q_t s_t + q_t^B b_t^m = n_t + d_t^m \tag{4}$$

and the law of motion of the net worth of mutual funds is

$$n_t = \left(R_t^k - R_{t-1}\right) Q_{t-1} s_{t-1} + \left(R_t^B - R_{t-1}\right) q_{t-1}^B b_{t-1}^m + R_{t-1} n_{t-1} \tag{5}$$

where d_t^m represents the money that investment banks borrow from the central bank, Q_t is the real price of capital, q_t^B is the real price of long-term treasury bonds, R_t^k is the real return on equity, R_t^B is the real return on long-term treasury bonds, and R_{t-1} is the short-term real interest rate.

Since each period a proportion of $1-\theta^m$ of mutual funds exit the financial market, at the aggregate level, if there were no startups, the aggregate net worth would shrink and concentrate among the luckiest mutual funds. To ensure the stability of the aggregate net worth of the mutual funds, I assume that each period sees the entry of new mutual fund companies into the financial market, the total net worth of which is a fraction φ of the aggregate effective assets $\phi_t N_{t-1}$. Aggregating the net worth of the surviving mutual funds provides the law of motion for the aggregate net worth, such that

$$N_t = \theta^m \left[\left(R_t^k - R_{t-1}\right) Q_{t-1} S_{t-1} + \left(R_t^B - R_{t-1}\right) q_{t-1}^B B_{t-1}^m + R_{t-1} N_{t-1}\right] + \varphi \phi_t N_{t-1}$$

where $N_t = \int n_{j,t} dj$.

Solving the optimization problem 2 yields the non-arbitrage condition $\lambda^b E_t \beta \Omega_{t,t+1} \left(R_{t+1}^k - R_t\right) = E_t \beta \Omega_{t,t+1} \left(R_{t+1}^B - R_t\right)$. This demonstrates that the excess return on firm equity should be same as the excess return on long-term bonds, adjusted by the collateral constraint parameter λ^b.

Define the effective leverage ratio of the mutual fund at time t as $\phi_t = \frac{Q_t s_t + \lambda^b q_t^B b_t^m}{n_t}$. Due to the collateral constraint 3, the mutual funds cannot borrow as much as they want, which implies an upper boundary for the leverage ratio

$$\phi_t \leq \overline{\phi} = \frac{E_t \left[\beta \Omega_{t,t+1} R_t\right]}{\lambda^v - E_t \left[\beta \Omega_{t,t+1} \left(R_{t+1}^k - R_t\right)\right]} \tag{6}$$

and the non-negative relatively the effective multiplier λ_t[12] such that

$$\lambda_t = \max \left\{0, 1 - \frac{\zeta_t}{\lambda^v \phi_t}\right\} \tag{7}$$

where ζ_t is the marginal return of net worth and $\Omega_{t,t+1}$ is the augmented SDF.

Equation 6 suggests that the larger the borrowing cost $\beta \Omega_{t,t+1} R_t$ is, the less money mutual

[12] Denote ν^λ is the Lagrange multiplier of corollary constraint 3 and λ_t is called the effective multiplier as $\lambda_t = \frac{\upsilon_t}{1+\upsilon_t}$

funds want to borrow. Therefore, the lower boundary of the leverage is as the mutual funds have smaller borrowing demand and lower liability volume[13]. Additionally, the larger the excess return of firm equity $E_t\left[\beta\Omega_{t,t+1}\left(R^k_{t+1}-R_t\right)\right]$ is, the more valuable the mutual funds are, and V_t becomes larger. Consequently, the mutual funds' collaterals become more valuable, and they can borrow more from the central bank, which expands the upper boundary of the leverage ratio.

2.3 Intermediate good producer

Intermediate good producers use a production function with constant returns to scale $Y^m_t = A^{\mathrm{TFP}}_t\left(U_t\xi_t K_{t-1}\right)^\alpha L_t^{1-\alpha}$ to produce intermediate goods at time t. K_{t-1} is the total capital used; L_t is the labor demand; A^{TFP}_t is the technology level; and U_t is the capital utilization rate, which is determined at time t[14]. ξ_t is the effective capital shock, and all the old capital after production will be $\xi_t K_{t-1}$[15]. Therefore, the law of motion of capital is

$$K_t = \xi_t K_{t-1} + I_t - \delta\left(U_t\right)\xi_t K_{t-1} \tag{8}$$

The depreciation rate $\delta\left(U_t\right)$ is a function of capital utilization U_t which is first-order convex and second-order semi-convex, such that $\delta'\left(\cdot\right) > 0$ and $\delta''\left(\cdot\right) \geq 0$[16][17].

The intermediate goods producers choose U_t and L_t to produce goods and pay wage costs, depreciation costs, and real fixed costs. They solve the problem

$$\Pi^f_t = \max_{U_t,L_t} P^m_t Y^m_t - W_t L_t - \delta\left(U_t\right)\xi_t K_{t-1} - \tau_{y^m} \tag{9}$$

where τ_{y^m} is the real fixed production cost that is refunded to equity holders and will be sent back to the investment bank to ensure the fixed cost is non-distortional[18]. This refund to the investment bank is to maintain the non-distortion of the fixed cost. Note that the real depreciation cost is $\delta\left(U_t\right)\xi_t K_{t-1}$ instead of $Q_t\delta\left(U_t\right)\xi_t K_{t-1}$, which isolates the inflation and price setting problems from capital fluctuation. Therefore, the gross equity return of the intermediate goods

[13]Rewrite equation 6 as $\overline{\phi} = \frac{1}{\frac{\lambda^v - E_t\left[\beta\Omega_{t,t+1}R^k_{t+1}\right]}{E_t\left[\beta\Omega_{t,t+1}R_t\right]}+1}$. Since $\lambda^v - E_t\left[\beta\Omega_{t,t+1}R^k_{t+1}\right] < 0$ a larger $E_t\left[\beta\Omega_{t,t+1}R_t\right]$ induces a lower $\overline{\phi}$.

[14]This form of production function is proposed by Greenwood et al. (1988) to generate endogenous depreciation rate and more fluctuating real rental rate of capital.

[15]This is directly followed from Merton (1973) who uses effective capital to generate a component within asset return which is a pure "shock" and is not connected with fundamental economy. Incorporating this effectiveness shock into model talking about unconventional monetary policy is firstly proposed by Gertler and Karadi (2011) and Gertler and Karadi (2018). which helps to increase the volatility of the capital return R^k and the power of financial accelerator.

[16]Incorporating this convex depreciation function can help to generate the investment response and equity return to shock which is argued by Jaimovich and Rebelo (2009); Christiano et al. (2014).

[17]The key element that is related to the response is the capital utilization elasticity of marginal depreciation $\frac{\delta''(u)u}{\delta'(u)}$. Therefore I use the depreciation function $\delta\left(u_t\right) = \overline{\delta} + \frac{Y_u}{1+v}u_t^{1+v} - \frac{Y_u}{1+v}$ where $Y_u = \alpha\frac{Y^m}{Qu^{1+\nu}K\xi}$.

[18]The only effect that τ_{y^m} does is to match R^k_t at steady state as Favilukis et al. (2017) and Bianchi and Mendoza (2018) did.

producer is

$$R_t^k = \frac{\left[\frac{\Pi_t^f + \tau_{ym}}{\xi_t K_{t-1}} + Q_t\right]\xi_t}{Q_{t-1}}$$

The intermediate goods producers do not have a balance sheet and are all owned by mutual funds, ensuring that the stock market clearing condition $S_t = K_t$ holds. The intertemporal optimization problem related to the state variable K_t is in fact solved by financial institutions, a setting that is equivalent to an environment where firms have a balance sheet and can accumulate net worth and solve the intertemporal problem through borrowing from the financial market[19].

2.4 Retailer and Final good producer

Retailers buy intermediate goods and differentiate them with monopolistic power, which allows them to freely set the price. Final goods producers use the differentiated goods from retailers to produce final goods via the standard CES function $Y_t = \left[\int_0^1 Y_{jt}^{(\sigma_p-1)/\sigma_p} dj\right]^{\sigma_p/(\sigma_p-1)}$ in a competitive market.

Retailers set the retail goods price based on the strategy proposed by Calvo (1983) such that

$$\max_{P_{jt}^*} E_t \sum_{\tau=0}^{\infty} (\theta\beta)^{\tau} \Lambda_{t,t+\tau} \left[\frac{P_{jt}^*}{P_{t+\tau}} \prod_{k=1}^{\tau} \Pi_{t+k-1}^{\gamma_p} - P_{t+\tau}^m\right] Y_{jt+\tau}$$
$$\text{s.t. } Y_{jt} = \left(\frac{P_{jt}}{P_t}\right)^{-\sigma_p} Y_t \quad (10)$$

Denote μ_t as the price dispersion which is defined as

$$\mu_t = \int_0^1 \left(\frac{P_{jt}}{P_t}\right)^{-\sigma_p} dj = (1-\theta)\left(\frac{\Pi_t^*}{\Pi_t}\right)^{-\sigma_p} + \theta\left(\frac{\Pi_{t-1}^{\gamma_p}}{\Pi_t}\right)^{-\sigma_p} \mu_{t-1}$$

Therefore, equation 10 can be written as $Y_t^m = \mu_t Y_t$.

2.5 Capital producers

Capital producers use final goods to produce physical capital and bear production costs according to the function $f(I_{n,\tau}, I_{n,\tau-1})$, which helps to match the investment response with respect to monetary policy[20]. Capital producers may earn profits, which are ultimately transferred to households via lump-sum transfers.

The capital producers maximize their present discounted real profit by choosing the net

[19] Carlstrom et al. (2012), Coenen et al. (2018) and Sims et al. (2022) use this type of setting. Carceles-Poveda and Coen-Pirani (2010) did a deeper investigation on this equivalence.

[20] This is firstly argued by Christiano et al. (2005).

investment this period such that

$$\max E_t \sum_{\tau=t}^{\infty} \beta^{\tau-t} \Lambda_{t,t+\tau} \left\{ (Q_\tau - 1) I_{n,\tau} - f(I_{n,\tau}, I_{n,\tau-1})(I_{n,\tau} + I_{ss}) \right\}$$

$$\text{s.t. } f(I_{n,\tau}, I_{n,\tau-1}) = \frac{\psi_I}{2} \left(\frac{I_{n,\tau} + I_{ss}}{I_{n,\tau-1} + I_{ss}} - 1 \right)^2$$

where $I_{n,\tau}$ is the net investment after depreciation such that $I_{n,\tau} = I_t - \delta(U_t)\xi_t K_{t-1}$.

This closes the supply market of capital and pins down the capital price as a convex function of investment, which follows

$$Q_t = 1 + \frac{\psi_I}{2} \left(\frac{I_{n,t} + I_{ss}}{I_{n,t-1} + I_{ss}} - 1 \right)^2 + \psi_I \left(\frac{I_{n,t} + I_{ss}}{I_{n,t-1} + I_{ss}} - 1 \right) \frac{I_{n,t} + I_{ss}}{I_{n,t-1} + I_{ss}} - E_t \beta \Lambda_{t,t+1} \psi_I \left(\frac{I_{n,t+1} + I_{ss}}{I_{n,t} + I_{ss}} - 1 \right) \left(\frac{I_{n,t+1} + I_{ss}}{I_{n,t} + I_{ss}} \right)^2 \quad (11)$$

2.6 Central Bank and Government

Either households or mutual funds can borrow money D_t^i from the central bank at time t and then pay the gross interest rate R_t at time $t+1$. Meanwhile, they also buy long-term treasury bondsB_t^{cb} and earn a gross return of $1 + \rho q_{t+1}^B$, realized at time $t+1$, which is paid by the government at a geometric decay rate ρ[21], following the budget constraint 13.

The government issues the long-term treasury bonds B_t^g at time t and funded its budget constraint by seigniorage T_t^s from central bank follows

$$T_t = T_t^s - \frac{(1 + \rho q_t^B)}{\Pi_t} B_{t-1}^g + q_t^B B_t^g \quad (12)$$

$$T_t^s + D_t^h - R_{t-1} D_{t-1}^h + D_t^m - R_{t-1} D_{t-1}^m = \frac{(1 + \rho q_t^B)}{\Pi_t} B_{t-1}^{cb} - q_t^B B_t^{cb} \quad (13)$$

where

$$B_t^g = B_t^{cb} + B_t^m = 0 \quad (14)$$

$$B_t^m = \int b_{i,t}^m di \quad (15)$$

$$D_t^h = -h^{\text{nHtM}} b_t^{\text{nHtM}} \quad (16)$$

where m represents mutual fund companies and h represents households. Further, I assume the supply of long-term bonds from the treasury department is constant at zero, which helps isolate the effect of monetary policy. I can simplify the equations 12 and 13 under the zero bond supply

[21] This type of modeling long-term bonds is followed by Woodford (2001)

assumption to

$$T_t + D_t^h - R_{t-1}D_{t-1}^h + D_t^m - R_{t-1}D_{t-1}^m = q_t^B B_t^m - \frac{\left(1+\rho q_t^B\right)}{\Pi_t} B_{t-1}^m \tag{17}$$

Moreover, the government is also in charge of providing income subsidies to hand-to-mouth households, which are funded by the labor income tax τ following the clearing condition $\tau w_t L_t = \left(h^{\text{pHtM}} + h^{\text{wHtM}}\right) \Theta_t^{\text{HtM}}$.

During the non-ZLB episode, the central bank stabilizes the economy via conventional monetary policy such that it directly sets the nominal interest rate $\mathcal{R}_t$ following the Taylor rule

$$\mathcal{R}_t = \max\left\{\mathcal{R}_{t-1}^{\theta_r} \mathbb{E}_t \left[\left(\frac{\Pi_{t+1}}{\Pi}\right)^{\theta_\pi} \left(\frac{Y_t}{Y}\right)^{\theta_y}\right]^{1-\theta_r} \gamma_t^{MP}, 1\right\} \tag{18}$$

where γ_t^{MP} is the monetary policy shock. Variables without time subscripts denote the steady-state values of corresponding variables. The real interest rate is pinned down by the nominal interest rate $R_t = \frac{\mathcal{R}_t}{\Pi_{t+1}}$.

In addition to conventional monetary policy, the central bank can also implement unconventional monetary policy by increasing its holdings of long-term treasury bonds. Since the supply of treasury bonds is fixed, as stated in equation 14, controlling the central bank's holdings of long-term bonds is equivalent to controlling the holdings of long-term bonds by mutual funds. Therefore, I set the QE policy rule[22] as

$$\frac{B_t^m}{\overline{B^m}} = \frac{B_{t-1}^m}{\overline{B^m}}^{\theta_r^{QE}} \left[\left(\frac{\Pi_{t+1}}{\Pi}\right)^{\theta_\pi^{QE}} \left(\frac{Y_t}{Y}\right)^{\theta_y^{QE}}\right]^{1-\theta_r^{QE}} \gamma_t^{QE} \tag{19}$$

Following Cui and Sterk (2021) and Sims et al. (2022) I assume the money used to implement QE policy is funded by household with lump-sum tax

$$q_t^B B_t^m - D_t^h = \overline{T}_{\text{cb}}$$

3 The power of QE: Decomposition

Utilizing the baseline model introduced in the previous section, I decompose the power of QE to expand output into two channels: the liquidity channel and the interest rate channel, presenting tractable analytical results in this section. By calculating the partial derivatives of output with respect to the derivatives of bond prices and bond values, I demonstrate how different QE mechanisms—either direct (balance sheet effect, pecuniary effect) or indirect (wage income

[22] Since $B_t^{cb} + B_t^m = 0$ where B_t^m is opposite to B_t^{cb}, the coefficient o reaction should be negative such that $\theta_\pi^{QE} < 0, \theta_y^{QE} < 0$.

general equilibrium effect, redistribution effect)—can be synthesized into these two channels, and they are distributed symmetrically or asymmetrically. Some mechanisms exist only in one channel, while others appear in both channels.

3.1 Transmission mechanism

What we can observe after the central bank implemented unconventional monetary policy is that $\Delta q_t^B B_t^m$ becomes negative since the central bank holds more long-term treasury bonds. At the same time, Δq_t^B is positive as the long-term yield decreases due to increased demand for the long-term bonds. The endpoint of unconventional monetary policy is that ΔY_t is positive, thus stimulating output, which has been widely investigated empirically[23]. There are several paths we can take from the starting point (negative $\Delta q_t^B B_t^m$ and positive Δq_t^B) to the endpoint (positive ΔY_t), which I have summarized in Table 1.

Table 1: QE Decomposition

QE effect	Liquidity channel	Interest rate channel
Supply side	liquidity easing	pecuniary easing
Demand side	redistribution effect	substitution effect

3.1.1 Liquidity channel

The liquidity channel states that unconventional monetary policy stimulates the economy by injecting liquidity into the market. In this sense, the liquidity that the central bank injects is just $\left|\Delta q_t^B B_t^m\right|$, as the central bank provides this amount of money to financial institutions in exchange for the long-term treasury bonds that the financial institutions previously held. Financial institutions will use this liquidity to invest in the supply side (firms) to induce more investment and production. In other words, unconventional monetary policy *crowds out* the holding of long-term bonds by financial institutions, which forces them to readjust their investment portfolios and put more investment on the production side to stimulate the economy[24].

Financial friction ensures that this liquidity injection is effective so that financial institutions will indeed use this extra liquidity to invest rather than to pay off their debts or accumulate new net worth. From equations 6 and 7, we can understand that if financial frictions did not exist or if the financial institutions were not in constrained states, it would be suboptimal for financial

[23] The output-stimulation effect of unconventional monetary policy is identified by Baumeister and Benati (2012); Kapetanios et al. (2012); Stock and Watson (2012); Weale and Wieladek (2016); Wu and Xia (2016); Di Maggio et al. (2020); Bauer and Rudebusch (2014); Swanson and Williams (2014); Engen et al. (2015); Hesse et al. (2018) and is summarized by Kuttner (2018); Lombardi et al. (2018); Borio and Zabai (2018)

[24] It works like the increased government spending contemporaneously as it expands the output by more demand through which the government increases its expenditure even though it does nothing more and drops this expenditure into the sea. However this supply-side stimulation is persistent because capital is a state variable and complementary to labour which will generate a long-lasting effect.

institutions to spend this liquidity on investing in equities. When ϕ_t is not fixed at the upper boundary, it is optimal for financial institutions to decrease their debt D_t^m and increase their net worth N_t^m, as the leverage ratio is not bound, giving financial institutions the propensity to decrease their debt. This partially explains why quantitative easing *does not always work*, for instance, in Japan. This effectiveness shows that financial friction is a double-edged sword for the economy and the central bank. During normal times, it causes the underinvestment problem because financial institutions cannot borrow as much as they want to invest, so the capital level is, in fact, below the optimal level. However, during the ZLB period, it contributes to the effectiveness of unconventional monetary policy by ensuring that the monetary policy works to stimulate the economy and decrease the real interest rate.

Additionally, financial frictions can also amplify the power of unconventional monetary policy as they may act as an accelerator for economic activities, which is referred to as the financial accelerator[25]. If the leverage ratio and net worth were fixed so that there was no endogenous portfolio adjustment, the $\left|\Delta q_t^B B_t^m\right|$ amount of crowded long-term bonds would ultimately generate a $\lambda^b \left|\Delta q_t^B B_t^m\right|$ amount of new investment in equity, given the leverage ratio is $\phi_t = \frac{Q_t s_t + \lambda^b q_t^B b_t^m}{n_t}$. The endogenous leverage ratio will further enlarge the effect, which is the heart of the financial accelerator such that the financial market responds more to shocks compared to the production side. This overresponse amplifies the shock's effect and induces the economy to become more fluctuant.

In addition to the stimulation generated by financial institutions, there will always be the general equilibrium effect to which the demand side contributes. Take the standard Euler equation $u_t' = \beta R_t E_t u_{t+1}'$ as an example. The real rental rate of capital R_t^k is closely linked with the real interest rate R_t as more investment will result in more production and higher inflation Π_{t+1}. Because unconventional monetary policy is implemented at the ZLB when the nominal interest rate is fixed, R_t will decrease, as driven by the Fisher equation. The decreased real interest rate will encourage households to consume more, following the Euler equation. The money they use to consume is the extra wage income paid by intermediate goods producers as there is more demand for final goods.

Moreover, apart from the standard Euler equation mechanism, the demand-side general equilibrium effect will be further magnified by heterogeneous households. If there were some households who were financially constrained so that their consumption did not follow the Euler equation, the general equilibrium effect would be amplified as these hand-to-mouth households consumed all the increased wage income, generating a strong feedback loop to the production sector. In addition to this Keynesian-cross effect, there is also a redistribution effect amplifying the power of unconventional monetary policy, akin to the redistribution channel analyzed by Auclert (2019) and Luetticke (2021). It is the real credit borrowed from households that the central bank uses to buy long-term treasury bonds from financial institutions. Non-hand-to-mouth

[25] This is first proposed by Bernanke et al. (1999)

households pay the cost of unconventional monetary policy, whereas they only earn some of the profit prompted by economic stimulation. This redistribution mechanism enlarges the effect of unconventional monetary policy as it redistributes wealth from non-hand-to-mouth households to hand-to-mouth households, who have a larger marginal propensity to consume.

3.1.2 Interest rate channel

The unconventional monetary policy not only decreases the holding of long-term treasury bonds by financial institutions but also twists the term yield and decreases the long-term interest rate. A positive Δq_t^B will also stimulate the economy, albeit through a different mechanism, the interest rate channel. The element related to long-term bonds within the budget constraint of the financial institutions is $(1+\rho^B q_t^B)B_{t-1}^m - q_t^B B_t^m$, which indicates that at time t, financial institutions hold $(1+\rho^B q_t^B)B_{t-1}^m$ amount of market-value-based long-term bonds. In addition to the bonds they have held since the last period, they choose B_t^m to take to the next period for which they spend $q_t^B B_t^m$ of money to buy. Even though there is no liquidity injected by the central bank as $q_t^B B_t^m$ is fixed, the economy is still stimulated because financial institutions can still obtain some liquidity from the bonds they have held since the last period, as now the price of long-term bonds is higher. After this pecuniary easing, financial institutions will invest more in corporate equity, leading to more investment and production. The non-arbitrage condition $\lambda^b E_t \beta \Omega_{t,t+1}\left(R_{t+1}^k - R_t\right) = E_t \beta \Omega_{t,t+1}\left(R_{t+1}^B - R_t\right)$ inspires financial institutions to invest more in the production sector as well, because the expected return of long-term bonds drops as $R_{t+1}^B = \frac{1+\rho^B q_{t+1}^B}{q_t^B}$. Similarly, the financial friction will strengthen the interest rate channel, although the response may not be as large as with the liquidity channel since there is no liquidity injection and all the amplification comes from endogenous leverage decisions.

Likewise, unconventional monetary policy can also stimulate the economy on the demand side via the interest rate channel, yet it operates on a completely different principle. Unconventional monetary policy changes the returns and portfolio of financial institutions, which subsequently varies the return on illiquid assets as the financial institutions are also suppliers of the illiquid asset. By examining equation 5, we can see that the return of financial institutions will increase because R_t^B and R_t^k will increase. The increased price of long-term treasury bonds initially drives up the return on long-term bonds this period (but drives down the expected return next period, which is also the current yield for this period[26]). Due to the non-arbitrage condition and financial friction, the financial institutions increase their investment in equity, which in turn increases R_t^k as more investment leads to a higher capital price, although it also decreases the expected return of equity next period as more capital means a lower real rental rate.

Proposition 1. *Denote the PE effect of unconventional monetary policy to illiquid asset return is*

[26]It is worth to notice that the yield to maturity this period $q_t^{-1} + \rho^B - 1$ also decreases.

$\frac{\partial \widehat{R}_t^a}{\partial \widehat{q}_t^B}\Big|^{PE}_{\widehat{q}_t^B+\widehat{B}_t^m=q^B+B^m}$ *such that*

$$\frac{\partial \widehat{R}_t^a}{\partial \widehat{q}_t^B}\Bigg|^{PE}_{\widehat{q}_t^B+\widehat{B}_t^m=q^B+B^m} = \frac{(1-\theta^m)\, R^B B^m q^B}{RN^h}\frac{\rho^B q^B}{1+\rho^B q^B} > 0$$

Proposition 1 shows that the partial equilibrium effect of unconventional policy on illiquid asset returns via the liquidity channel is positive within a first-order linear system. Meanwhile, from equation 72, we know that if we also consider the capital and goods markets, the above effect will be larger. The financial accelerator further expands this effect through endogenous leverage and a feedback loop. This inflated illiquid asset return may work positively or negatively on the demand side, which is determined by the illiquid asset substitution effect of households. A larger $\widehat{R}_t^a$ will affect neither wealthy nor poor hand-to-mouth households since the consumption of the former is governed only by the real interest rate and the latter is governed only by budget constraints (but they do not hold any illiquid assets). Conversely, a larger $\widehat{R}_t^a$ will affect the wealthy hand-to-mouth households as they indeed hold the illiquid asset, and their consumption is governed by the budget constraint. However, whether this effect is positive or negative is regulated by the substitution effect of wealthy hand-to-mouth households. They can consume most of the increased asset return and save a small amount of it; alternatively, they can also save most of the increased asset return or even decrease their consumption to further invest in the illiquid asset because now the illiquid asset return is higher[27]. I assume that the illiquid asset withdrawal X_t^i is fixed in the baseline model for tractability, so that the PE effect of unconventional monetary policy on the demand side is muted. Nonetheless, the general equilibrium effect of unconventional monetary policy on the demand side always persists.

3.2 Transmission magnitude and Complementary effect

In the previous two subsections, I argued that unconventional monetary policy works through liquidity and interest rate channels on the supply and demand sides separately, yet they do not only influence the economy separately. They also have a positive cross effect through which unconventional monetary policy stimulates the economy. This complementary effect is not unique to unconventional monetary policy; Bilbiie et al. (2022) first proposed this complementary effect for conventional monetary policy, which they called the *multiplier of multiplier* effect. The proposition below decomposes the stimulation power of unconventional monetary policy into the liquidity channel and interest channel.

Proposition 2. *When the price and depreciation rate is fixed, the contemporaneous effect of unconventional monetary policy on output can be decomposed to liquidity and interest rate channel such that*

[27] This substitution effect is propelled by the marginal propensity to take risk which was first argued by Kekre and Lenel (2021).

$$\left.\frac{\partial \widehat{Y}_t}{\partial\left(\widehat{q}_t^B+\widehat{B}_t^m\right)}\right|_{\widehat{q}_t^B=q^B}=-\frac{\frac{1}{h^n}-1+\frac{\lambda^b}{\phi}+\varphi_1^m\frac{\varphi_1^h}{Th^n}}{C^n\frac{\psi}{\sigma}+\frac{(1-\tau)WL}{h^n}+\varphi_1^m\varphi_4^h}q^B B^m=-\varphi_L q^B B^m \tag{20}$$

$$\left.\frac{\partial \widehat{Y}_t}{\partial \widehat{q}_t^B}\right|_{\widehat{q}_t^B+\widehat{B}_t^m=q^B+B^m}=\frac{\rho}{C^n\frac{\psi}{\sigma}+\frac{(1-\tau)WL}{h^n}+\varphi_1^m\varphi_4^h}q^B B^m=\varphi_R q^B B^m \tag{21}$$

where $\varphi_1^h=h_C^w T_{C^w}+h_C^p T_{C^p}$, $\varphi_2^h=h_C^w\Theta_{C^w}^T+h_C^p\Theta_{C^p}^T$, $\varphi_3^h=h_C^w\frac{C^n}{C^w}+h_C^p\frac{C^n}{C^p}$, $\varphi_4^h=Y_C-\delta K_C-\varphi_2^h+\frac{\varphi_1^h(1-\tau)WL}{h^nT}+\left(h_C^n+\varphi_3^h\right)\frac{\psi}{\sigma}$ and $\varphi_1^m=\left(N^h\frac{1-\theta^m+\theta^m\eta}{\theta^m}\lambda\frac{R^k}{R^k-R}\frac{1}{KQ+\Pi^f}-1+\frac{1}{\phi}\right)\frac{\varphi_I}{\delta}(1-\beta\Lambda)-\left(1-\frac{1}{\phi}\right)Q$.

Firstly, let us focus on the non-cross effect at the supply and demand sides. The heterogeneous household on the demand side is involved in the denominator for either the liquidity channel or the interest rate channel. The existence of idiosyncratic shocks and precautionary saving augments the power of unconventional monetary policy through hand-to-mouth households as C^n is now smaller than its value in the representative agent model. Nevertheless, the income effect is now attenuated as $\frac{(1-\tau)WL}{h^n}$ is larger ($h^n<1$) because only the wealthy household is earning the wage income, and the GE effect cannot work through this[28]. The phenomenon that these two effects offset each other is just the difference between average consumption and cross-sectional consumption dispersion, as argued by Debortoli and Galí (2022). As they argued, if we allow the hand-to-mouth households to work and the hand-to-mouth community is generated endogenously by financial friction, as Kaplan et al. (2018) did, instead of purely by assumption, the attenuation effect $\frac{(1-\tau)WL}{h^n}$ would vanish. This GE effect will emerge in both channels as the stimulation starts from the production sector where financial institutions invest and passes to the household sector via the income effect.

In addition to the GE effect on the denominator of equations 20 and 21 through which either the liquidity channel or the interest rate channel works, there is another redistribution effect on the numerator for the liquidity channel, which I call the *redistribution credit* effect. The money that the central bank uses to buy long-term treasury bonds from financial institutions is funded by borrowing from households, specifically, wealthy non-hand-to-mouth households. The non-hand-to-mouth household pays 1 unit of money to the central bank, which passes it to financial institutions in exchange for 1 unit of market-value-based long-term treasury bonds. However, there are two channels through which this 1 unit of money returns to households: one is through the net worth of the financial institutions, and the other is through the liabilities of the financial institutions, which ultimately flows into lump-sum tax transfers because they fund their liabilities by borrowing from the central bank. Hence, there is a redistribution effect such that

[28]In my model, there is no cyclical redistribution problem as proposed by Broer et al. (2020). The offset problem they argued is based on the assumption that the wealthy household does not supply labor and captures all the equity return from firms. In this type of setting, the effect of countercyclical wage income on hand-to-mouth households will offset the effect of procyclical markup on non-hand-to-mouth households. Therefore, the total effect will become very small, but only the redistribution effect exists.

$\frac{1}{h^n} - 1$ amount of wealth is redistributed from non-hand-to-mouth households to hand-to-mouth households. Along with this redistribution credit effect, there is an extra term $\frac{\lambda^b}{\phi}$ that comes from the portfolio adjustment effect through which the financial institutions' portfolios are adjusted because the long-term treasury bonds they previously held are crowded out by the central bank.

Aside from the non-cross effects at the supply and demand sides, unconventional monetary policy works complementarily at both sides. Proposition 2 shows that $\varphi_1^m \varphi_4^h$ is one of the complementary effects at the supply and demand sides. φ_4^h is the component related to the demand side, and it takes effect through the general equilibrium $Y_C - \delta K_C$. This effect is amplified by the heterogeneous household through lump-sum tax transfers φ_1^h, unemployment insurance φ_2^h , and consumption dispersion φ_3^h. φ_1^m is the component related to the supply side, which determines the direction of complementarity between heterogeneous households and financial friction. The proposition below decomposes the components of complementarity in the financial market into two effects: the redistribution return effect and the redistribution wealth effect.

Proposition 3. *The complementary component of stimulation effect at supply side can be further decomposed as*

$$\varphi_1^m = \underbrace{\left(N^h \frac{1-\theta^m+\theta^m\eta}{\theta^m} \lambda \frac{R^k}{R^k - R} \frac{1}{KQ + \Pi^f} - 1 + \frac{1}{\phi} \right) \frac{\varphi_I}{\delta} (1 - \beta\Lambda)}_{\text{redistribution return}} - \underbrace{\left(1 - \frac{1}{\phi}\right) Q}_{\text{redistribution wealth}}$$

The redistribution return effect comes from the contemporaneous asset return, which is expanded by unconventional monetary policy since more demand for output entails a larger real rental rate. However, this inflated return is entirely captured by wealthy non-hand-to-mouth households, which does not contribute to the GE stimulation on the demand side as their consumption is driven by the real interest rate in the Euler equation. The redistribution wealth effect is similar to the redistribution credit effect in the liquidity channel but occurs at a different position of the asset. The redistribution credit effect relates to the credit from the central bank to wealthy households, from whom the central bank borrows 1 unit of money (to buy long-term bonds) but only refunds $\frac{1}{h^n}$ back to them, which I denote as D^h in the model part. The redistribution wealth effect relates to the asset and net worth of financial institutions whose capital holding increases by 1 unit of money in value because of the stimulation from the central bank, but only $\frac{1}{\phi}$ is owned by wealthy households. The remaining $1 - \frac{1}{\phi}$ portion is ultimately owned by hand-to-mouth households.

It is obvious that these two effects offset each other. The redistribution return effect abates the amplification ability of consumption inequality, as a large proportion of the excess return is earned by the wealthy household. Meanwhile, the redistribution wealth effect contributes in another direction, as not all the stimulated boom of assets belongs to the wealthy household, despite there being $\frac{1}{\phi}$ amount. The complementary component on the supply side works as the *multiplier of multiplier*, and the financial friction here acts as the *multiplier of the multiplier of* *multiplier* because the leverage ratio ϕ is greater than 1.

The corollary below implies that only when the redistribution wealth effect overwhelms the redistribution return effect will the complementary effect between the financial accelerator and heterogeneous households further magnify the effect of unconventional monetary policy.

Corollary 1. *The contemporaneous effect of unconventional monetary policy to output is magnified by complementarity between demand side and supply side as long as capital price at steady state is not too small.*

Proof. It is straightforward to prove that as long as Q is not too small, φ_1^m will be negative. The effect through the interest rate channel will be larger, as shown in equation 21. I can also derive the relationship that $\frac{\varphi_1^h}{Th^n} < \varphi_4^h$ from their definitions, so that a negative φ_1^m will also increase the effect through the liquidity channel, as indicated in equation 20. $\square$

Given the calibration in next section figure 1 reveals the coefficient value of liquidity channel φ_L and interest rate channel φ_R, after varying different parameters. 1a shows that complementary effect grows stronger along the process through which redistribution wealth effect surpassing redistribution return effect. 1b shows that the stimulation effect via liquidity channel expends, accompanying the severity of consumption inequality. Nevertheless the effect through interest rate channel becomes more and more silent because the complementary effect is negative to the inequality at steady state where I pin down the capital price to 1. 1c illustrates the amplification power of financial accelerator through the random exit and entry. When the possibility to exit is higher (θ^m is smaller), the financial institutions will respond to unconventional monetary policy more intensively, ergo larger stimulation power. At steady state the ratio of stimulation power between these two channels, $\frac{\varphi_L}{\varphi_R}$, is 1.26 , a number which is close to the empirical result. The empirical stimulation power ratio $\frac{\Delta\widehat{Y}_L}{\Delta\widehat{Y}_R}$ to the media impulse response at figure 5 is 1.46, averaging along the time line.

Given the calibration in the next section, figure 1 reveals the coefficient values of the liquidity channel φ_L and the interest rate channel φ_R, after varying different parameters. 1a shows that the complementary effect grows stronger through the process in which the redistribution wealth effect surpasses the redistribution return effect. 1b shows that the stimulation effect via the liquidity channel expands, accompanying the severity of consumption inequality. Nevertheless, the effect through the interest rate channel becomes increasingly muted because the complementary effect is negative to inequality at steady state where I pin down the capital price to 1. 1c illustrates the amplification power of the financial accelerator through random exit and entry. When the possibility to exit is higher (θ^m is smaller), the financial institutions will respond to unconventional monetary policy more intensively, ergo larger stimulation power. At steady state, the ratio of stimulation power between these two channels, $\frac{\varphi_L}{\varphi_R}$, is 1.26, a number which is close to the empirical result. The empirical stimulation power ratio $\frac{\Delta\widehat{Y}_L}{\Delta\widehat{Y}_R}$ to the median impulse response at figure 5 is 1.46, averaging along the timeline.

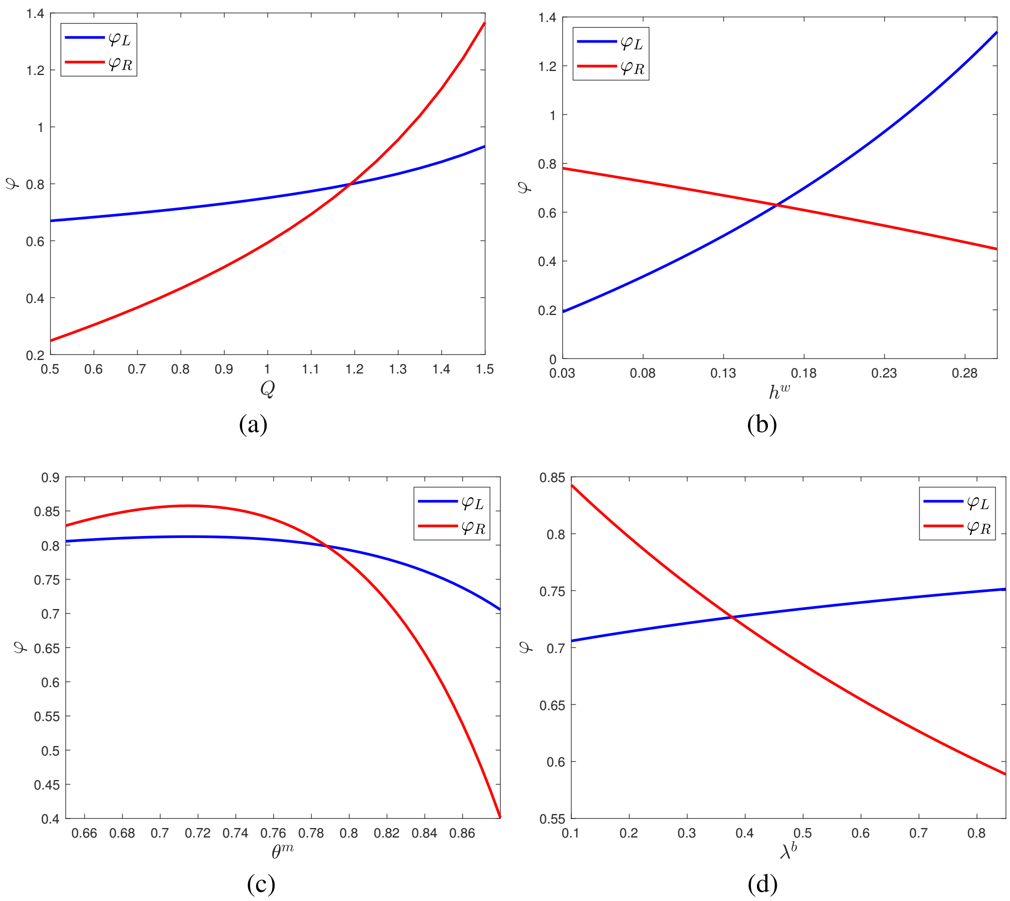


Figure 1: Complementary effect

4 Quantitative experiment

The argument in the last section about the two channels through which unconventional monetary policy works is restricted and conservative because of unrealistic assumptions such as fixed prices and depreciation rates. Therefore, I solve the model quantitatively in this section to clarify the channels through which unconventional monetary policy operates. After calibrating the model and then comparing the quantitative results with the estimated empirical impulse responses, I conduct some counterfactual experiments to quantitatively exhibit the power of unconventional monetary policy on different channels.

4.1 Calibration

The calibration process is standard and supported by literature, consisting of four main parts: household, financial institutions, central bank, and production sector. I only elaborate on the first three parts and relegate the production sector to an appendix, where parameters come from the literature. All of the key parameters in these three sectors are summarized in table 2.

4.1.1 Household

The parameters in the utility function are standard; I set the intertemporal elasticity of substitution σ to 2, as well as the inverse Frisch elasticity ψ. The disutility of labor κ is set to 1 for convenience and conventionality. The discount factor is set to 0.98, which is a common value to match the precautionary saving motive in heterogeneous agent literature, as done by Auclert et al. (2021). The real interest rate of liquid bonds is 2% at an annual rate, which is close to the interest rate before the Great Recession. The shares of different types of households come from the literature and are identified by Kaplan et al. (2014). The shares of poor hand-to-mouth households h^{pHtM}, wealthy hand-to-mouth households h^{wHtM}, and non-hand-to-mouth households h^{nHtM} are 0.121, 0.192 and 0.687, respectively. The probability of a household entering into a hand-to-mouth state from a non-hand-to-mouth state p^{EU} is 0.044, targeted by the monthly inflow rate of unemployment at 1.5%, as collected by the Current Population Survey. Meanwhile, I assume that conditional on becoming a hand-to-mouth household, a non-hand-to-mouth household uniformly becomes either a poor hand-to-mouth or wealthy hand-to-mouth household following their relative group size, so that $p^{\text{pHtM}} = p^{EU}\frac{h^{\text{pHtM}}}{h^{\text{pHtM}}+h^{\text{wHtM}}} = 0.017$ and $p^{\text{wHtM}} = 0.027$. Although the total illiquid asset withdrawal is determined at steady state by the market clearing condition, the distribution of the withdrawal is not. To calibrate the distribution of illiquid asset withdrawal, I adjust the withdrawal ratio $\frac{X_t^{\text{wHtM}}}{X_t^{\text{nHtM}}}$ to match the income-over-output ratio of wealthy hand-to-mouth households[29].

4.1.2 Financial Institutions and Central Bank

The relative collateral constraint between equity and long-term bonds, λ^b, is 0.83 as proposed by Gertler and Karadi (2018) and Karadi and Nakov (2021). The gross collateral constraint, λ^ν, targets the leverage ratio at 6. The geometric decay rate of long-term treasury bonds is set to target the duration of long-term bonds as 10 years. I calibrate the proportion of startup companies, φ, to match the public debt-to-GDP ratio. Further, I calibrate the possibility of mutual fund survival at 0.85 to match the quarterly equity return $R^k = 1.0256$, which I derive from the stock market as operating income after depreciation (Compustat item OIADPQ). This survival rate is slightly lower than that in previous literature, which does not match the equity return and assumes the excess return as zero. However, I target the return because my focus is on the financial market's contribution to the power of QE, in which the asset return and the non-arbitrage condition between different markets are important.

The parameters that influence monetary policy are directly taken from Cui and Sterk (2021) and Sims et al. (2022), where the central bank does not adjust the holding of long-term bonds endogenously, and $\theta_\pi^{QE} = \theta_y^{QE} = 0$, because this helps to decompose the different mechanisms

[29] The data of $\frac{c_t^{\text{wHtM}}}{Y_t}$ is calculated based on the average annual income of wealthy hand-to-mouth household which is roughly 40000 dollars in US at 2010 price and estimated by Kaplan et al. (2014). Then based on the population and nominal GDP at 2010 we can get $\frac{c^{\text{wHtM}}}{Y} = 0.82$.

through which QE works[30].

Table 2: Key Parameter Values

Parameter	**Value**	**Description**
β	0.98	Discount factor
τ	0.25	Labor income tax
ρ	0.995	Geometric decay rate of long-term bonds
θ^m	0.85	Exist rate of mutual funds
λ^b	0.83	Relative financial friction slackness
λ^ν	0.36	Absolute financial friction
h^{HtM}	0.313	Share of hand-to-mouth household
h^{nHtM}	0.687	Share of non hand-to-mouth household
h^{wHtM}	0.192	Share of wealthy hand-to-mouth household
h^{pHtM}	0.121	Share of poor hand-to-mouth household
p^{EU}	0.044	Possibility go from nHtM to HtM
p^{UE}	0.097	Possibility go from HtM to nHtM
$h^{\text{wHtM}\|\text{HtM}}$	0.613	Share of wealthy hand-to-mouth conditional on HtM
$h^{\text{pHtM}\|\text{HtM}}$	0.387	Share of poor hand-to-mouth conditional on HtM
X	0.55	Total illiquid asset withdrawing

4.2 Quantitative result

I first conducted a simulation based on my baseline model to show that the model successfully replicates some critical macroeconomic facts during the implementation of a large-scale asset purchasing policy. I fixed the nominal interest rate at the steady state to mimic the ZLB condition and considered a 1 percentage point unexpected unconventional monetary policy shock, γ_t^{QE}. Figure 2 shows the impulse response to the unconventional monetary policy shock. The central bank spends real money borrowed from households to buy long-term treasury bonds from financial institutions, which expands the demand for long-term treasury bonds and increases the bond price by 0.64 percent. This crowds out the long-term treasury bonds' holding of financial institutions and inspires them to invest more in the stock market, which triggers substantial physical investment. The expansion in capital demand elevates the capital price and spurs the real economy through both partial and general equilibrium. The crowded long-term treasury bonds are replaced by investment, generating a stimulation in output through partial equilibrium. In addition to the physical investment, the labor market is also activated and the households increase

[30]The goals of this paper is not to find the optimal monetary policy rule.

their consumption as their labor income jumps. This jump, compared with that in physical capital, generates stimulation in output through general equilibrium. Combining the partial and general equilibrium effects, the output experiences a maximized jump of 0.656 percent corresponding to a 0.485 percent jump in consumption. The increased demand for the final good encourages retailers to set higher prices, pushing up inflation and generating a boom.

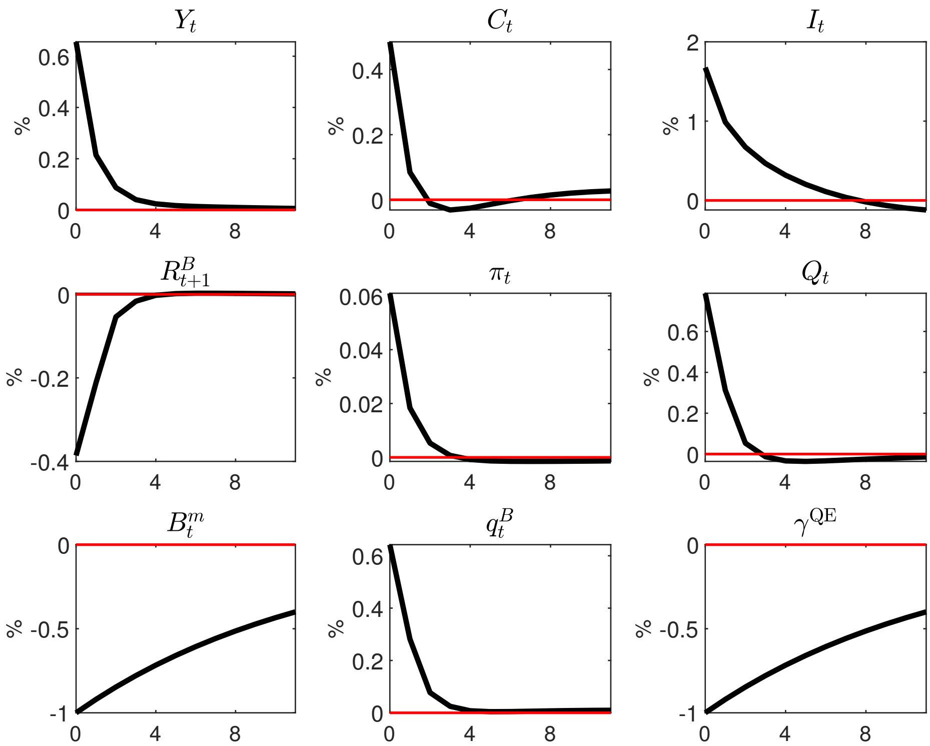


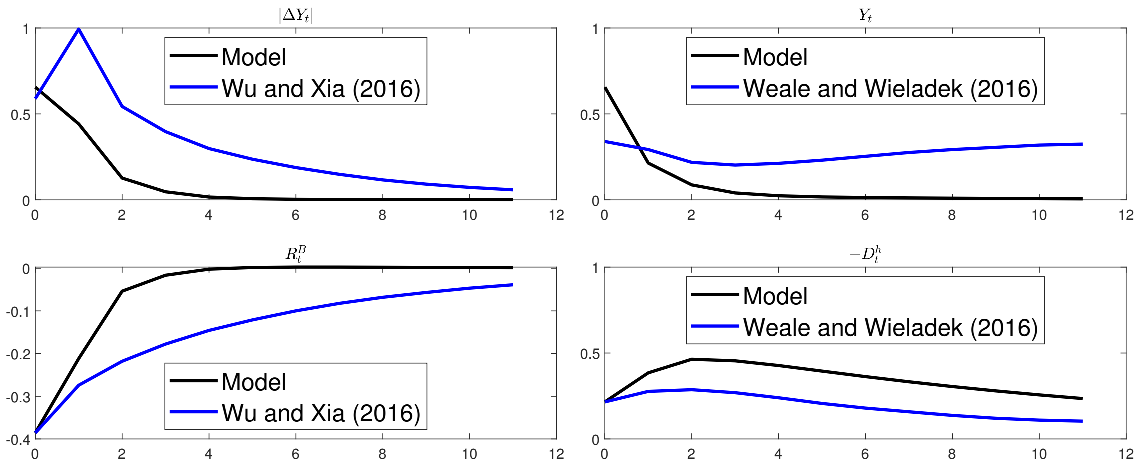
Figure 2: IRF to an unexpected expansionary unconventional monetary policy shock

In addition to inflation in the capital and goods markets, unconventional monetary policy decreases the interest rate on long-term bonds R^B_{t+1}, which acts as the shadow rate of the economy. The model shows that a 0.38 percent decline in the shadow rate stimulates a 0.64 percent increase in real output, a finding supported by the empirical research of Wu and Xia (2016), which showed a 0.59 percent output effect with the same decline in the shadow rate. Figure 3a displays a comparison of the impulse response between the empirical identification and the model's simulation, revealing that the model indeed can help us to unveil the mechanisms of stimulation of unconventional monetary policy[31]. In addition to empirical identification focusing on the shadow rate, other scholars have tried to use accumulated asset purchasing announcements to identify the power of unconventional monetary policy[32]. The model yields a

[31] Here I compare the absolute value of first-order difference in output after the first period. The reason is that the VAR result of Wu and Xia (2016) generates a permanent shock in output which implies all the ΔY_t in their model are positive. However in my model the QE shock is transitory so that all the ΔY_t is negative as long as there is no hump shape or over shooting. Therefore I plot the absolute value in figure 3a otherwise the line will be symmetric along the x axis.

[32] Gambacorta et al. (2014); Panizza et al. (2016); Weale and Wieladek (2016); Hesse et al. (2018) did these work and used the accumulated asset purchasing announcement as the unconventional monetary policy shock to identify the QE effect to macroeconomic.

0.76 percent jump in GDP at the peak, given spending on monetary policy equivalent to 1 percent of GDP at an annual rate, a figure that aligns closely with the empirical findings[33] of Weale and Wieladek (2016). Figure 3bshows the association between the model and the empirical results of Weale and Wieladek (2016) by rescaling the money that the central bank used to buy long-term treasury bonds[34]. The comparison between empirical identification results and the simulation results generated by the model provides the rationality of the model, affirming that it indeed represents what happened in reality quantitatively. Thus, it is credible to use this model to conduct several counterfactual experiments to disclose how unconventional monetary policy works via the interest rate and liquidity channels.



(a) Comparing to Wu and Xia (2016). (b) Comparing to Weale and Wieladek (2016)

Figure 3: Model vs Empirical. IRFs are in percentage level.

Proposition 2 shows that the pecuniary effect is pivotal in the interest rate channel, which is regulated by the geometric decay rate ρ. Accordingly, the QE effect via the liquidity channel could be isolated as long as $\rho = 0$, a condition where the interest rate channel is silent and only the liquidity channel is left[35]. Additionally, the effect of unconventional monetary policy on the supply and demand sides can be transparently revealed by replacing the heterogeneous households with a representative household[36] or weakening the capacity of the financial accelerator[37]. Figure 4 illustrates the stimulation power of unconventional monetary policy with different channels given the same influence on the shadow rate[38] at time 0. The solid black line represents

[33] By the identification scheme 2 they yield a 0.77 percent jump in GDP at the peak.

[34] The reason why relative effect of GDP is smaller, comparing to 0.76 vs 0.77 at the peak, is that the result of Weale and Wieladek (2016) is based on monthly frequency but the model is based on quarterly frequency (though the peak point is comparable as their meaning is percentage deviation from steady sate where two results have different steady state). To convert their result to quarterly frequency I need to take sum of GDP but take mean of the accumulated asset purchasing. This mutated the stimulation effect of their work.

[35] In experiment I set ρ to a infinit small number, 10^{-8}, instead of exactly at 0 for computational convenience.

[36] The interest rate at steady state changes to $\frac{1}{\beta}$ at RA scenario because the precautionary saving motive disappears under representative-household setting but the Euler equation still holds.

[37] To be consistent with the argument in last section I undermine the effect of financial accelerator by setting the possibility of surviving θ_m close to 1.

[38] Since at different situations the amount of long-term bonds and price are different at steady state, it is

the combination of all mechanisms through which unconventional monetary policy operates, and it is the largest one in output because all four effects in table 1 are positive. After shutting down the interest rate channel, we can observe that the stimulation effect on output becomes smaller, as now the stimulation only acts via the liquidity channel, despite the extent of the drop not being large. This is shown by the dashed blue line, which lies below the baseline model almost throughout the timeline. The real GDP increases by 0.4 percent at the peak through the liquidity channel, which is roughly 60 percent of the effect in the baseline model. The gap in output between these two lines implies that the ratio of the two stimulation channels, $\frac{\varphi_L}{\varphi_R}$, is approximately[39] 1.5, which is slightly larger than the analytical result of 1.3 (derived under ultra assumptions) and exactly the same as the empirical result in the next section. The difference between the baseline and liquidity models suggests that unconventional monetary policy stimulates the economy across these two channels, which have the same direction yet different diameters. The liquidity channel helps to engender immense stimulation relative to the interest rate channel because the central bank injects liquidity into the market and boosts more demand for physical capital when the financial institutions are constrained and lack liquidity.

Furthermore, the two channels, interest rate and liquidity, are connected with the supply and demand sides of the economy, into which financial institutions flow funds. Figure 4 shows that neither the supply side nor the demand side plays a significant and central role through which unconventional monetary policy undertakes stimulation. It is the complementarity between the supply (financial friction) and demand (heterogeneous household) sides that makes monetary policy effective again during the ZLB period[40]. The dashed pink line delineates the refined demand side effect through the HANK model, where only heterogeneous households exist, yet no financial accelerator. Similarly, the dashed green line highlights the pure supply side effect through the RANK model, where only financial friction prevails, yet no heterogeneous household, regarding the baseline model[41]. Both of these models generate a slight stimulation on output and pertain to a higher shadow rate compared to the baseline model which combines them. The vast disparity ascertains a new discrepancy between conventional and unconventional

inappropriate to compare with the same influence on the amount of crowded long-term treasury bonds or the money spent by central bank. Therefore shadow rate is the best one to connect the power of different channels compared to other state variables, response of which is determined endogenously and influenced by their steady states. Moreover, matching shadow rate can also tie the model to the VAR identification in next section.

[39] I approximate the effect of interest rate channel $\varphi_R = 0.6562 - 0.3944 = 0.2618$ by assuming the effects of these two channels are separable and additive. Therefore the ratio of the effect of two channels $\frac{\varphi_L}{\varphi_R}$ is approximated with $\frac{0.3944}{0.2618} = 1.5$.

[40] The "complementarity" here is different with the "complementarity" in proposition 2. In previous section the complementarity $\varphi_1^m \varphi_4^h$ is the effect conditional on the existence of heterogeneous household and financial accelerator. The complementarity in figure 4 is the difference between "together" effect and "separate" effect. In other words to identify the complementarity in figure 4 I also change the HA effect and financial accelerator effect in proposition 2. $C^n \frac{\psi}{\sigma} + \frac{(1-\tau)WL}{h^n}$, $\frac{1}{h^n} - 1$, φ_1^h, φ_2^h and other factors changed when I identified the complementarity in figure 4.

[41] The RANK model is modified by endogenous illiquid asset withdrawing following the extension experiments in appendix of Cui and Sterk (2021) because RANK model with exogenous withdrawing cannot be solved as Blanchard-Kahn condition is not satisfied. The extension in appendix B.1 implies that the response of RANK model in figure 4 is a conservative result as endogeneity amplifies the power of unconventional monetary policy.

monetary policy. Even though both heterogeneous households and financial friction can magnify the effect of conventional monetary policy separately, they hardly take effect on unconventional monetary policy alone. However, they can considerably amplify the power of unconventional monetary policy as long as they are connected and work together.

Additionally, the stimulation direction of the two channels, interest rate and liquidity, is stable and will not become opposite in any subset of the economy, neither the supply sector nor the demand sector. The effect of unconventional monetary policy in the model with a representative household, without the interest rate channel, is depicted by the dashed red line in figure 4. The stimulation effect of the model with the liquidity channel is slightly smaller than that with both channels in output, although all of them peak modestly as there is no financial accelerator here. This discrepancy verifies the argument in the last section that financial friction and recession generate a scarcity of liquidity. The two channels become effective by providing liquidity through direct injection or a pecuniary effect, despite not being augmented by heterogeneous households and general equilibrium. However, the heterogeneous household and redistribution effect still play a role here by spawning a positive effect of the interest rate channel on consumption, contrasting with the negative effect in the RANK model in figure 4. Both of these channels require the financial institutions to borrow money from the central bank to invest, which results in a higher leverage ratio. The debts borrowed by financial institutions are funded by lump-sum taxes and ultimately come from households. Therefore, the larger the stimulation effect on output, the smaller the stimulation on consumption will be, since now the redistribution effect is absent.

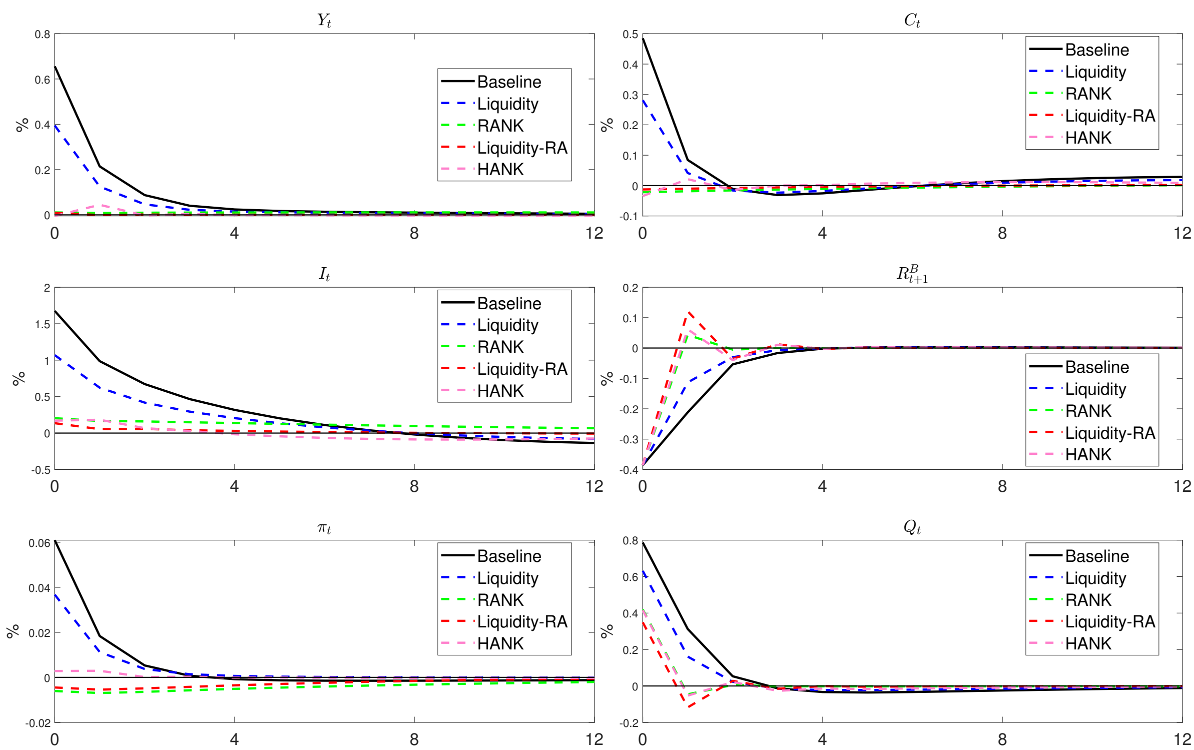


Figure 4: Decomposition of unconventional monetary policy in different channels

5 Empirical Evidence

In the last two sections, I analytically discussed how unconventional monetary policy stimulates the economy through two channels and quantitatively illustrated their relative effects. In this section, I provide empirical evidence to show that the analytical discussion and quantitative results are tractable and reasonable. Before introducing the new instrumental variable and identification results, I first explain the new method and a new type of intuitive inequality constraint used to conduct Bayesian estimation on a VAR model with multiple instrumental variables. Then, using a new instrumental variable, along with an existing one, I disentangle the effects of unconventional monetary policy into the two channels and provide empirical evidence to support my argument.

5.1 Methodology

I use a new IV estimation process to estimate a VAR to identify the liquidity supply effect and interest rate expectation effect of monetary policy. The formal one links to liquidity channel and the later one links to interest rate channel. To estimate the monetary policy effect to the economy, I need to estimate the following reduce-form var model

I use a new IV estimation process to estimate a VAR to identify the liquidity supply effect and interest rate expectation effect of monetary policy. The former links to the liquidity channel and the latter links to the interest rate channel. To estimate the monetary policy effect on the economy, I need to estimate the following reduced-form VAR model

$$Y_t = \sum_{j=1}^{p} A_j Y_{t-j} + B\boldsymbol{\varepsilon}_t$$

where $\boldsymbol{\varepsilon}_t$ is an iid shock and I can normalize its covariance matrix to identity such that $E\left[\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t'\right] = I$.

It is easy to estimate A_j while we can only get $E\left[u_t u_t'\right] = BB'$ where

$$u_t = B\boldsymbol{\varepsilon}_t$$

It is impossible to identify B from BB' and we need further $\frac{n(n+1)}{2}$ restrictions to identify it.

Stock and Watson (2012) and Mertens and Ravn (2013) proposed a new method that introduced proxies variable to help identifying the shock effect on economy. They introduced k new variable which satisfies

$$E\left[m_t\boldsymbol{\varepsilon'}_{1t}\right] = \Phi \tag{22}$$

$$E\left[m_t\boldsymbol{\varepsilon'}_{2t}\right] = 0 \tag{23}$$

where m_t is a k-by-1 vector and I take partition on $\boldsymbol{\varepsilon}_t$ such that $\boldsymbol{\varepsilon}_t = \begin{bmatrix} \boldsymbol{\varepsilon}_{1t}' & \boldsymbol{\varepsilon}_{2t}' \end{bmatrix}'$.

Introducing m_t helps us to identify the ε_{1t} effect on the economy because Φ provides more information about ε_{1t} given m_t. Unfortunately, identifying ε_{2t} cannot receive any help as $E\left[m_t \boldsymbol{\varepsilon}'_{2t}\right] = 0$ and I need other restrictions on B if fully identifying B is the objective. However, I am only interested in the monetary policy effect and only need to identify the first k columns of $\boldsymbol{B}$, as Gertler and Karadi (2015) did. While I still need more restrictions on B to identify ε_{1t} since identifying Φ wastes some degree of freedom. As long as $k > 1$ is held and Φ was fully identified, the first k column of B cannot be fully identified. Mertens and Ravn (2013) added several linear restrictions helping to identify the effect of ε_{1t}. In this paper, one of my contributions is that conversely, I propose a modified method that imposes restrictions on Φ and leaves the first k columns of B free to identify.

Write B into partition

$$
\begin{aligned}
B &= \begin{bmatrix} b_{11} & b_{12} & \boldsymbol{b}_{13} \\ b_{21} & b_{22} & \boldsymbol{b}_{23} \\ \boldsymbol{b}_{31} & \boldsymbol{b}_{32} & \boldsymbol{b}_{33} \end{bmatrix}_{n \times n} \\
&= \begin{bmatrix} \beta_1 & \beta_2 \end{bmatrix}
\end{aligned}
$$

$$
\boldsymbol{\beta}_1 = \begin{bmatrix} \beta_{11} & \boldsymbol{\beta}'_{21} \\ \beta_{12} & \boldsymbol{\beta}'_{22} \end{bmatrix}'_{n \times 2}
$$

Combining with equation 22 and 23, it is easy to yield

$$
\Phi \boldsymbol{\beta}'_1 = \Sigma_{mu'} \tag{24}
$$

Furthermore denote $s_{11} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ the covariance matrix of coefficient Φ can be written as

$$
\Phi\Phi' = \Sigma_{mu'_1} \left(\Sigma'^{-1}_{mu'_1} s_{11} s'_{11} \right)^{-1} \tag{25}
$$

where $\Sigma_{mu'_1}$ is the first $k \times k$ elements of $\Sigma_{mu'}$; $s^{-1}_{11} s'^{-1}_{11}$ can be estimated from equation 82. I demote related derivative process of equation 25 to appendix.

Then, taking the Cholesky decomposition on $\Phi\Phi'$ yields the lower triangular matrix Φ_{tr}. Φ will be identified up to the rotation matrix Q such that $\Phi = \Phi_{tr} Q$. Note that Q is an orthogonal matrix such that $Q \in \mathcal{O}(n)$. Following Caldara and Herbst (2019) and Arias et al. (2018), I develop an algorithm using the Metropolis-Hastings sampler across Haar measure space to draw Q from $Q \big| Y, X, M, B, u, \Sigma$. Since Baumeister and Hamilton (2015) argued that the standard method, which uses a uniform distribution on reduced-form parameters, causes a flat likelihood problem, I use a Bayesian method to estimate the reduced-form parameter Φ and $\boldsymbol{\beta}'_1$, which is developed by Caldara and Herbst (2019), Braun et al. (2020), Arias et al. (2021) and Giacomini et al. (2021).

Any $\widehat{\Phi} = \Omega_{tr}Q$ that satisfies the restrictions will give us a fully identified $\boldsymbol{\beta}_1'$ through equation 24, which is what I am interested in.

Alternatively, I also conduct some robustness checks on the estimation. I tried to impose off-diagonal zero restrictions and lower triangle restrictions on Φ. Meanwhile, I also tried to use a uniform prior and standard Bayesian estimation to estimate Φ. To estimate $\boldsymbol{\beta}_1'$, I use a frequentist method. All detailed explanations and results of the robustness checks are shown in Appendix D.

Considering the possibility of plausibly exogenous variables[42], I also impose an inequality constraint, an idea analogous to the $S(\phi, Q)$ inequality proposed by Giacomini et al. (2021). Here, I assume that

$$\rho\left(m_{it}\varepsilon_{it}\right)^2 > \rho\left(m_{it}\varepsilon_{jt}\right)^2 , \forall i \neq j \tag{26}$$

This inequality is intuitive and meaningful that implies the instrument explain much of the information of structural shock at the same row relative to the structural shock at other row. Given the definition of $\rho\left(m_{it}\varepsilon_{jt}\right)$

This inequality is intuitive and meaningful, implying that the instrument explains much of the information about the structural shock in the same row relative to the structural shock in other rows. Given the definition of $\rho\left(m_{it}\varepsilon_{jt}\right)$

$$\rho\left(m_{it}\varepsilon_{jt}\right)^2 = \frac{\operatorname{cov}\left(m_{it}\varepsilon_{jt}\right)^2}{\sigma^2_{m_{it}}\sigma^2_{\varepsilon_{jt}}}$$

and the fact that I normalize the structural shock ε_t to a standard normal distribution, equation 26 can be simplified to

$$\operatorname{cov}\left(m_{it}\varepsilon_{it}\right)^2 > \operatorname{cov}\left(m_{it}\varepsilon_{jt}\right)^2$$

Furthermore, because of the equation 24, I can write this inequality to

Furthermore, because of equation 24, I can write this inequality as

$$F\left(\Phi_{tr}, Q; \gamma\right) \equiv \operatorname{diag}\left\{\left(\Phi_{tr}Q\right) \circ \left(\Phi_{tr}Q\right)\begin{bmatrix} 1 & -\gamma_2 \\ -\gamma_1 & 1 \end{bmatrix}\right\} > 0$$

where γ_i represents the weakness[43] of the instrumental variables m_i. A valid instrumental variable m_i should satisfy $\gamma_i \geq 1$, which I assume will be satisfied throughout this paper. Based on the assumption of $\gamma_i \geq 1$, the domain of γ should be $[1, \infty]$. $\gamma_i = 1$ denotes the slackest restriction on the power of the instrumental variable m_i that explains the shock ε_i. It only needs to correlate with ε_i slightly more than with ε_j. $\gamma_i = \infty$ denotes the strongest restriction, which requires that the instrumental variable m_i is not correlated with ε_j at all and is only correlated

[42] Conley et al. (2012) detailedly discussed this type of proxy variables.

[43] The larger γ_i is, the stronger related instrument is that can be used to explain ε_i.

with ε_i. It is straightforward to write γ_i as $\kappa_i = 1 - \frac{1}{\gamma_i}$, where $\kappa_i \in [0, 1]$, which is easier to understand. $\kappa = 0$ represents no restriction[44] and $\kappa = 1$ represents the strongest restriction. Therefore, the above restriction can be written as

$$F(\Phi_{tr}, Q; \rho) \equiv \text{diag}\left\{ (\Phi_{tr}Q) \circ (\Phi_{tr}Q) \begin{bmatrix} 1 & -\frac{1}{1-\kappa_2} \\ -\frac{1}{1-\kappa_1} & 1 \end{bmatrix} \right\} > 0 \tag{27}$$

5.2 Bayesian estimation and result

Following Gertler and Karadi (2015), I use four variables to estimate the monetary policy effect on the economy, such that $Y_t = \begin{bmatrix} r_t & cpi_t & y_t & \Delta_t \end{bmatrix}$. r_t is the market yield on 2-Year U.S. Treasury Securities. cpi_t is the logarithmic Consumer Price Index. y_t is the logarithmic industrial production. Δ_t is the excess bonds premium, which is estimated through the methods proposed by Gilchrist and Zakrajšek (2012). I write the two instrumental variables as $m_t = \begin{bmatrix} m_{1t} & m_{2t} \end{bmatrix}'$ and use conventional high-frequency identification to construct related instrumental variables with respect to monetary policy shock. m_{1t} is the surprises in the current month's fed funds futures (FF1) contract price during the day when the Treasury Department announces a new issue of treasury bonds/notes. m_{2t} is the surprises in FF1 contract price during the day when the FOMC meeting is held and an FOMC announcement is released. This approach is widely used by scholars who analyze monetary policy using the changed futures contract price on FOMC announcement days[45]. Similar to an FOMC announcement, the US Department of the Treasury will disclose their intended amount of treasury bonds they plan to issue on the related announcement day. In contrast to the FOMC announcement, which shows the targeted short-term interest rate or federal funds rate, announcements issued by the Treasury Department only provide information related to the nominal face value of the bond they want to issue. The interest rate or yield to maturity is determined by an auction held several days later. Given that the central bank imposes monetary policy through the liquidity channel by open-market operations and buying long-term treasury bonds, the purely increased treasury bond supply will be isomorphic to decreased treasury bond holdings of the central bank, as long as general equilibrium has always been reached. Therefore, the surprises in the future federal funds rate contract price is a valid and appropriate instrumental variable to correlate with the pure liquidity effect of monetary policy since the only extra information treasury bond announcements provide is just the liquidity amount[46]. More detailed illustrations about data can be found in Appendix C.

Figure 5 shows the estimation result without imposing any constraints, such that $\kappa_1 = \kappa_2 = 0$.

---

[44]Here the word "no" means as long as the instruments is valid the estimation of $\mathbf{\Phi}$ is acceptable.

[45]A large series of literature discuss monetary policy through this high-frequency identification such as Kuttner (2001); Gürkaynak et al. (2004); Bernanke and Kuttner (2005); Hamilton (2008); Campbell et al. (2012); Hanson and Stein (2015); Nakamura and Steinsson (2018)

[46]Respectively the only information provided by FOMC is the targeted interest rate (federal fund rate) instead of the liquidity amount. Though during the QE period central bank indeed provided liquidity information in some statements, I do the robustness check which excludes QE period and it yields the same result.

The left column (5a) shows the IRF to the liquidity shock, and the right column (5b) shows the IRF to the interest rate shock. Both shocks are normalized to generate a 1 percentage point interest rate deviation at period 0.

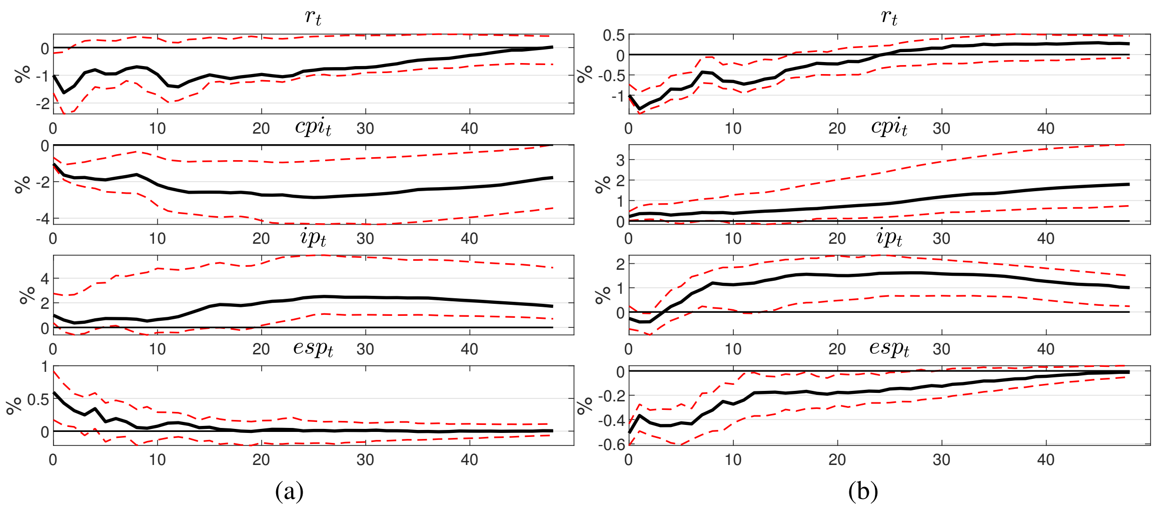


Figure 5: IRF to unconventional monetary policy under liquidity and interest rate channel with 90% confidence band.

Figure 5 sheds light on the power of unconventional monetary policy in the liquidity and interest rate channels empirically. A one percent increase in the 2-year Treasury bond rate[47], triggered by a contractionary unconventional monetary policy, will suppress output by 2.5% at the peak via the liquidity channel and by 1.6% at the peak via the interest rate channel. The empirical result justifies the argument that the liquidity channel is more powerful than the interest rate channel in stimulating output. To associate the empirical result with the quantitative result from the last section, I cannot directly use the above result since the long-term rate in figure 5 is the change of the yield-to-maturity rate, while the shadow rate in the model is the current-yield rate. Therefore, I use the same identification method but with 3-month U.S. Treasury Securities instead of 2-year to conduct the estimation again in the appendix. The consistency between my empirical and quantitative results substantiates my main contribution in this paper that the effect of the liquidity channel on the stimulation power of QE on output is approximately 1.5 times larger than that of the interest rate channel.

6 Conclusion

I introduce household heterogeneity and a financial accelerator into a general equilibrium model to analyze how unconventional monetary policy impacts the economy through the liquidity and

[47]Following Gertler and Karadi (2015) I select the independent variable, interest rate, as 2-years treasury bond. Furthermore, 2-years is the only suitable variable since I need some variation in interest rate to pin down the response of output. However the definition of liquidity channel represents the change in output without effect on long-term bonds rate so that the 5-years and 10-years treasury bonds are not suitable.

interest rate channels. I first decompose these two channels and discuss how their effects are governed by supply and demand sides through financial friction, pecuniary easing, redistribution of wealth and income, and complementarity, under a knife-edge condition where prices and depreciation rates are fixed. After this discussion, I carefully calibrate the model and show that the liquidity channel is quantitatively 1.5 times larger than the interest rate channel. I also demonstrate that the complementary effect between household heterogeneity and financial friction is pivotal in determining the power of unconventional monetary policy through the two channels. Finally, I use an IV-VAR model to demonstrate that the discussion and arguments are reasonable both qualitatively and quantitatively.

However, my research still has some drawbacks awaiting future exploration. The model is a simple DSGE with only three types of households, so that the mass and consumption decisions of hand-to-mouth households are all determined by exogenous assumptions. Moreover, the setting of financial friction and the endogenous leverage ratio are relatively simplistic compared to reality, where there exists heterogeneity within financial institutions and their exit and entry are endogenously regulated.

References

Arias, Jonas E, Juan F Rubio-Ramírez, and Daniel F Waggoner. 2018. "Inference based on structural vector autoregressions identified with sign and zero restrictions: Theory and applications." *Econometrica* 86 (2): 685–720.

Arias, Jonas E, Juan F Rubio-Ramírez, and Daniel F Waggoner. 2021. "Inference in Bayesian Proxy-SVARs." *Journal of Econometrics* 225 (1): 88–106.

Auclert, Adrien. 2019. "Monetary policy and the redistribution channel." *American Economic Review* 109 (6): 2333–67.

Auclert, Adrien, Bence Bardóczy, Matthew Rognlie, and Ludwig Straub. 2021. "Using the sequence-space Jacobian to solve and estimate heterogeneous-agent models." *Econometrica* 89 (5): 2375–2408.

Bauer, Michael, and Glenn D Rudebusch. 2014. "The signaling channel for Federal Reserve bond purchases." *International Journal of Central Banking*.

Baumeister, Christiane, and Luca Benati. 2012. "Unconventional monetary policy and the great recession: Estimating the macroeconomic effects of a spread compression at the zero lower bound."Technical report, Bank of Canada.

Baumeister, Christiane, and James D Hamilton. 2015. "Sign restrictions, structural vector autoregressions, and useful prior information." *Econometrica* 83 (5): 1963–1999.

Bayer, Christian, Ralph Lütticke, Lien Pham-Dao, and Volker Tjaden. 2019. "Precautionary savings, illiquid assets, and the aggregate consequences of shocks to household income risk." *Econometrica* 87 (1): 255–290.

Bernanke, Ben S, Mark Gertler, and Simon Gilchrist. 1999. "The financial accelerator in a quantitative business cycle framework." *Handbook of macroeconomics* 1 1341–1393.

Bernanke, Ben S, and Kenneth N Kuttner. 2005. "What explains the stock market's reaction to Federal Reserve policy?" *The Journal of finance* 60 (3): 1221–1257.

Bianchi, Javier, and Enrique G Mendoza. 2018. "Optimal time-consistent macroprudential policy." *Journal of Political Economy* 126 (2): 588–634.

Bilbiie, Florin O. 2020. "The new keynesian cross." *Journal of Monetary Economics* 114 90–108.

Bilbiie, Florin O, Diego R Känzig, and Paolo Surico. 2022. "Capital and income inequality: An aggregate-demand complementarity." *Journal of Monetary Economics* 126 154–169.

Borio, Claudio, and Anna Zabai. 2018. “Unconventional monetary policies: a re-appraisal.” In *Research Handbook on Central Banking*, 398–444, Edward Elgar Publishing.

Braun, Robin, Ralf Brüggemann et al. 2020. *Identification of SVAR models by combining sign restrictions with external instruments*. GSDS-Graduate School of Decision Sciences, University of Konstanz.

Broer, Tobias, Niels-Jakob Harbo Hansen, Per Krusell, and Erik Öberg. 2020. “The New Keynesian transmission mechanism: A heterogeneous-agent perspective.” *The Review of Economic Studies* 87 (1): 77–101.

Caldara, Dario, and Edward Herbst. 2019. “Monetary policy, real activity, and credit spreads: Evidence from Bayesian proxy SVARs.” *American Economic Journal: Macroeconomics* 11 (1): 157–92.

Calvo, Guillermo A. 1983. “Staggered prices in a utility-maximizing framework.” *Journal of monetary Economics* 12 (3): 383–398.

Campbell, Jeffrey R, Charles L Evans, Jonas DM Fisher, Alejandro Justiniano, Charles W Calomiris, and Michael Woodford. 2012. “Macroeconomic effects of federal reserve forward guidance [with comments and discussion].” *Brookings papers on economic activity* 1–80.

Cantore, Cristiano, and Lukas B Freund. 2021. “Workers, capitalists, and the government: fiscal policy and income (re) distribution.” *Journal of monetary economics* 119 58–74.

Carceles-Poveda, Eva, and Daniele Coen-Pirani. 2010. “Owning capital or being shareholders: An equivalence result with incomplete markets.” *Review of Economic Dynamics* 13 (3): 537–558.

Carlstrom, Charles T, Timothy S Fuerst, and Matthias Paustian. 2012. “How inflationary is an extended period of low interest rates?”.

Carlstrom, Charles T, Timothy S Fuerst, and Matthias Paustian. 2017. “Targeting long rates in a model with segmented markets.” *American Economic Journal: Macroeconomics* 9 (1): 205–42.

Chang, Minsu, Xiaohong Chen, and Frank Schorfheide. 2021. “Heterogeneity and aggregate fluctuations.”Technical report, National Bureau of Economic Research.

Christiano, Lawrence J, Martin Eichenbaum, and Charles L Evans. 2005. “Nominal rigidities and the dynamic effects of a shock to monetary policy.” *Journal of political Economy* 113 (1): 1–45.

Christiano, Lawrence J, Roberto Motto, and Massimo Rostagno. 2014. “Risk shocks.” *American Economic Review* 104 (1): 27–65.

Coenen, Günter, Peter Karadi, Sebastian Schmidt, and Anders Warne. 2018. "The New Area-Wide Model II: an extended version of the ECB's micro-founded model for forecasting and policy analysis with a financial sector."

Conley, Timothy G, Christian B Hansen, and Peter E Rossi. 2012. "Plausibly exogenous." *Review of Economics and Statistics* 94 (1): 260–272.

Cui, Wei, and Vincent Sterk. 2021. "Quantitative easing with heterogeneous agents." *Journal of Monetary Economics* 123 68–90.

Debortoli, Davide, and Jordi Galí. 2022. "Idiosyncratic income risk and aggregate fluctuations."Technical report, National Bureau of Economic Research.

Di Maggio, Marco, Amir Kermani, and Christopher J Palmer. 2020. "How quantitative easing works: Evidence on the refinancing channel." *The Review of Economic Studies* 87 (3): 1498–1528.

Engen, Eric M, Thomas Laubach, and David Reifschneider. 2015. "The macroeconomic effects of the Federal Reserve's unconventional monetary policies."

Favilukis, Jack, Sydney C Ludvigson, and Stijn Van Nieuwerburgh. 2017. "The macroeconomic effects of housing wealth, housing finance, and limited risk sharing in general equilibrium." *Journal of Political Economy* 125 (1): 140–223.

Gambacorta, Leonardo, Boris Hofmann, and Gert Peersman. 2014. "The effectiveness of unconventional monetary policy at the zero lower bound: A cross-country analysis." *Journal of Money, Credit and Banking* 46 (4): 615–642.

Gertler, Mark, and Peter Karadi. 2011. "A model of unconventional monetary policy." *Journal of monetary Economics* 58 (1): 17–34.

Gertler, Mark, and Peter Karadi. 2015. "Monetary policy surprises, credit costs, and economic activity." *American Economic Journal: Macroeconomics* 7 (1): 44–76.

Gertler, Mark, and Peter Karadi. 2018. "Qe 1 vs. 2 vs. 3...: A framework for analyzing large-scale asset purchases as a monetary policy tool." *29th issue (January 2013) of the International Journal of Central Banking*.

Gertler, Mark, and Nobuhiro Kiyotaki. 2015. "Banking, liquidity, and bank runs in an infinite horizon economy." *American Economic Review* 105 (7): 2011–2043.

Giacomini, Raffaella, Toru Kitagawa, and Matthew Read. 2021. "Robust Bayesian inference in proxy SVARs." *Journal of Econometrics*.

Gilchrist, Simon, and Egon Zakrajšek. 2012. "Credit spreads and business cycle fluctuations." *American economic review* 102 (4): 1692–1720.

Gornemann, Nils, Keith Kuester, and Makoto Nakajima. 2016. "Doves for the rich, hawks for the poor? Distributional consequences of monetary policy." *Distributional Consequences of Monetary Policy (April 2016).*

Greenwood, Jeremy, Zvi Hercowitz, and Gregory W Huffman. 1988. "Investment, capacity utilization, and the real business cycle." *The American Economic Review* 402–417.

Guerrieri, Veronica, and Guido Lorenzoni. 2017. "Credit crises, precautionary savings, and the liquidity trap." *The Quarterly Journal of Economics* 132 (3): 1427–1467.

Gürkaynak, Refet S, Brian P Sack, and Eric T Swanson. 2004. "Do actions speak louder than words? The response of asset prices to monetary policy actions and statements." *The Response of Asset Prices to Monetary Policy Actions and Statements (November 2004).*

Hagedorn, Marcus, Jinfeng Luo, Iourii Manovskii, and Kurt Mitman. 2019. "Forward guidance." *Journal of Monetary Economics* 102 1–23.

Hamilton, James D. 2008. "Daily monetary policy shocks and new home sales." *Journal of Monetary Economics* 55 (7): 1171–1190.

Hanson, Samuel G, and Jeremy C Stein. 2015. "Monetary policy and long-term real rates." *Journal of Financial Economics* 115 (3): 429–448.

Harrison, Richard. 2017. "Optimal quantitative easing."

Hesse, Henning, Boris Hofmann, and James Michael Weber. 2018. "The macroeconomic effects of asset purchases revisited." *Journal of Macroeconomics* 58 115–138.

Jaimovich, Nir, and Sergio Rebelo. 2009. "Can news about the future drive the business cycle?" *American Economic Review* 99 (4): 1097–1118.

Kapetanios, George, Haroon Mumtaz, Ibrahim Stevens, and Konstantinos Theodoridis. 2012. "Assessing the economy-wide effects of quantitative easing." *The Economic Journal* 122 (564): F316–F347.

Kaplan, Greg, Benjamin Moll, and Giovanni L Violante. 2018. "Monetary policy according to HANK." *American Economic Review* 108 (3): 697–743.

Kaplan, Greg, Giovanni L Violante, and Justin Weidner. 2014. "The wealthy hand-to-mouth."Technical report, National Bureau of Economic Research.

Karadi, Peter, and Anton Nakov. 2021. "Effectiveness and addictiveness of quantitative easing." *Journal of Monetary Economics* 117 1096–1117.

Kekre, Rohan, and Moritz Lenel. 2021. "Monetary policy, redistribution, and risk premia."Technical report, National Bureau of Economic Research.

Krishnamurthy, Arvind, and Annette Vissing-Jorgensen. 2011. "The effects of quantitative easing on interest rates: channels and implications for policy."Technical report, National Bureau of Economic Research.

Kuttner, Kenneth N. 2001. "Monetary policy surprises and interest rates: Evidence from the Fed funds futures market." *Journal of monetary economics* 47 (3): 523–544.

Kuttner, Kenneth N. 2018. "Outside the box: Unconventional monetary policy in the great recession and beyond." *Journal of Economic Perspectives* 32 (4): 121–46.

Lombardi, Domenico, Pierre Siklos, and Samantha St. Amand. 2018. "A survey of the international evidence and lessons learned about unconventional monetary policies: Is a 'new normal'in our future?" *Journal of Economic Surveys* 32 (5): 1229–1256.

Luetticke, Ralph. 2021. "Transmission of monetary policy with heterogeneity in household portfolios." *American Economic Journal: Macroeconomics* 13 (2): 1–25.

McKay, Alisdair, Emi Nakamura, and Jón Steinsson. 2016. "The power of forward guidance revisited." *American Economic Review* 106 (10): 3133–58.

Mertens, Karel, and Morten O Ravn. 2013. "The dynamic effects of personal and corporate income tax changes in the United States." *American economic review* 103 (4): 1212–47.

Merton, Robert C. 1973. "An intertemporal capital asset pricing model." *Econometrica: Journal of the Econometric Society* 867–887.

Nakamura, Emi, and Jón Steinsson. 2018. "High-frequency identification of monetary nonneutrality: the information effect." *The Quarterly Journal of Economics* 133 (3): 1283–1330.

Panizza, Ugo, Charles Wyplosz et al. 2016. "The folk theorem of decreasing effectiveness of monetary policy: What do the data say?" In *17th J. Polak Annual Research Conference of the IMF*.

Rubio-Ramirez, Juan F, Daniel F Waggoner, and Tao Zha. 2010. "Structural vector autoregressions: Theory of identification and algorithms for inference." *The Review of Economic Studies* 77 (2): 665–696.

Sims, Eric R, Jing Cynthia Wu, and Ji Zhang. 2022. "Unconventional Monetary Policy According to HANK." *Available at SSRN*.

Sims, Eric, and Jing Cynthia Wu. 2021. "Evaluating central banks' tool kit: Past, present, and future." *Journal of Monetary Economics* 118 135–160.

Stock, James H, and Mark W Watson. 2012. “Disentangling the Channels of the 2007-2009 Recession.”Technical report, National Bureau of Economic Research.

Swanson, Eric T, and John C Williams. 2014. “Measuring the effect of the zero lower bound on medium-and longer-term interest rates.” *American economic review* 104 (10): 3154–85.

Uhlig, Harald. 2005. “What are the effects of monetary policy on output? Results from an agnostic identification procedure.” *Journal of Monetary Economics* 52 (2): 381–419.

Weale, Martin, and Tomasz Wieladek. 2016. “What are the macroeconomic effects of asset purchases?” *Journal of monetary Economics* 79 81–93.

Woodford, Michael. 2001. “Fiscal requirements for price stability.”

Wu, Jing Cynthia, and Fan Dora Xia. 2016. “Measuring the macroeconomic impact of monetary policy at the zero lower bound.” *Journal of Money, Credit and Banking* 48 (2-3): 253–291.

A Derivation steps and supplements to Model

A.1 Supplements to Model

In this section I provide some supplements to the model part with more detailed explanation and definition related to the model setting, as well as some important first-order conditions. The order of subsections are akin to section 2 from household to central bank and government.

A.1.1 Household

Since the hand-to-mouth household will consume all their income each period, the consumption of poor and wealth hand-to-mouth household is static such that

$$c_t^{pHtM} = \Theta_t^{\mathrm{HtM}} + T_t$$

and

$$c_t^{\mathrm{wHtM}} = X^{\mathrm{wHtM}} + \Theta_t^{\mathrm{HtM}} + T_t$$

Meanwhile as only the non hand-to-mouth household can supply the labour to firm, the supply function of labour will be pinned down by the first-order condition of non hand-to-mouth household

$$\frac{L_t}{h^{\mathrm{nHtM}}} = \left(-\frac{(1-\tau_l)\, w_t}{\kappa}\right)^{\frac{1}{\psi}} \left(c_t^{\mathrm{nHtM}}\right)^{-\frac{\sigma}{\psi}}$$

It is worth to notice that the distributional cyclicity problem between labour income and dividend income[48] in HANK model does not emerge here because only the non hand-to-mouth household provides labour into market.

A.1.2 Mutual funds

The budget constraint of the mutual fund 5 can be written as

$$n_t = R_t^k Q_{t-1} s_{t-1} - Q_t s_t + \frac{(1+\rho q_t^B)}{\Pi_t} b_{t-1}^m - q_t^B b_t^m - R_{t-1} d_{t-1}^m$$

Therefore I define the return of long-term treasury bonds earned by mutual fund as

$$R_t^B = \frac{(1+\rho^B q_t^B)}{q_{t-1}^B \Pi_t}$$

Meanwhile the value function of mutual fund W_t and V_t can be solved via guess and verify such that

$$W_t = \eta_t n_t$$

[48]This is firstly proposed by Broer et al. (2020) and further discussed by Cantore and Freund (2021).

$$V_t = \mu_t^s Q_t s_t + \mu_t^b q_t^B b_t^m + \zeta_t n_t$$

$$\zeta_t = E_t \beta \Omega_{t,t+1} R_t$$

$$\mu_t^s = E_t \beta \Omega_{t,t+1} \left(R_{t+1}^k - R_t \right)$$

$$\mu_t^b = E_t \beta \Omega_{t,t+1} \left(R_{t+1}^B - R_t \right)$$

A.1.3 Intermediate goods producer

The first-order conditions to maximization problem 9 are

$$P_t^m \alpha \frac{Y_t^m}{U_t} = \delta' \left(U_t \right) \xi_t K_{t-1}$$

and

$$P_t^m (1-\alpha) \frac{Y_t^m}{L_t} = W_t$$

Notice that the intermediate goods market is a complete market where producer earns zero profit and this pins down the real price of intermediate goods P_t^m.

A.1.4 Retailer and Final goods producer

The price setting problem 10 can be solved and rewritten into recursive formula

$$\Pi_t^* = \frac{\sigma_p}{\sigma_p - 1} \frac{F_t^p}{Z_t^p} \Pi_t$$

$$\Pi_t^* = \frac{P_t^*}{P_{t-1}}$$

$$F_t^p = P_t^m Y_t + \beta \theta E_t \Lambda_{t,t+1} \Pi_t^{-\gamma_p \sigma_p} \Pi_{t+1}^{\sigma_p} F_{t+1}^p$$

$$Z_t^p = Y_t + \beta \theta E_t \Lambda_{t,t+1} \Pi_t^{\gamma_p (1-\sigma_p)} \Pi_{t+1}^{\sigma_p - 1} Z_{t+1}^p$$

$$\Pi_t^{1-\sigma_p} = \theta \left(\Pi_{t-1}^{\gamma_p} \right)^{1-\sigma_p} + (1-\theta) \Pi_t^{*1-\sigma_p}$$

where Π_t^* is the inflation rate for those retailers who adjust their price at period t.

A.2 Log-linearization of Baseline Model

Household

$$\widehat{C}_t^w = \frac{T}{C^w} \widehat{T}_t + \frac{\Theta^T}{C^w} \widehat{\Theta}_t^T \tag{28}$$

$$\widehat{C}_t^p = \frac{T}{C^p} \widehat{T}_t + \frac{\Theta^T}{C^p} \widehat{\Theta}_t^T \tag{29}$$

$$-\sigma\widehat{C}_t^n = \widehat{R}_t - \sigma\left[\frac{p^E C^n}{\Sigma p^i C^{i(-\sigma)}}\widehat{C}_{t+1}^n + \frac{p^{EUw} C^w}{\Sigma p^i C^{i(-\sigma)}}\widehat{C}_{t+1}^w + \frac{p^{EUp} C^p}{\Sigma p^i C^{i(-\sigma)}}\widehat{C}_{t+1}^p\right] \tag{30}$$

$$\widehat{L}_t = \frac{1}{\psi}\widehat{W}_t - \frac{\sigma}{\psi}\widehat{C}_t^n \tag{31}$$

$$\widehat{C}_t = \frac{h^n}{C}\widehat{C}_t^n + \frac{h^w}{C}\widehat{C}_t^w + \frac{h^p}{C}\widehat{C}_t^p \tag{32}$$

Financial market

$$\widehat{\phi}_t = \frac{QK}{\phi N^h}\left(\widehat{Q}_t + \widehat{K}_t\right) + \frac{\lambda^b q^B B^m}{\phi N^h}\left(\widehat{q}_t^B + \widehat{B}_t^m\right) - \widehat{N}_t^h \tag{33}$$

$$\widehat{\Omega}_t = \widehat{\Lambda}_t + \frac{\theta^m}{1-\theta^m+\theta^m\eta}\widehat{\eta}_t \tag{34}$$

$$\widetilde{\lambda}_t = \frac{\zeta}{\phi\gamma^\lambda\lambda^\nu}\left(\widehat{\phi}_t + \widehat{\gamma}_t^\lambda - \widehat{\zeta}_t\right) \tag{35}$$

$$\widehat{\zeta}_t = \widehat{\Omega}_{t+1} + \widehat{R}_t \tag{36}$$

$$\widehat{\eta}_t = \widehat{\zeta}_t + \frac{1}{1-\lambda}\widetilde{\lambda}_t \tag{37}$$

$$\widehat{R}_t^k = \widehat{\xi}_t + \frac{Q}{Q+\frac{\Pi^f}{\xi K}}\widehat{Q}_t - \widehat{Q}_{t-1} + \frac{\Pi^f}{\xi KQ+\Pi^f}\left(\widehat{\Pi}_t^f - \widehat{\xi}_t - \widehat{K}_{t-1}\right) \tag{38}$$

$$\widehat{R}_t^B = \frac{\rho q^B}{1+\rho q^B}\widehat{q}_t^B - \widehat{\Pi}_t - \widehat{q}_{t-1}^B \tag{39}$$

$$\widehat{\Omega}_{t+1} + \frac{R^k}{R^k-R}\widehat{R}_{t+1}^k - \frac{R}{R^k-R}\widehat{R}_t = \frac{1}{\lambda}\widetilde{\lambda}_t + \widehat{\gamma}_t^\lambda \tag{40}$$

$$\widehat{\Omega}_{t+1} + \frac{R^B}{R^B-R}\widehat{R}_{t+1}^B - \frac{R}{R^B-R}\widehat{R}_t = \frac{1}{\lambda}\widetilde{\lambda}_t + \widehat{\gamma}_t^\lambda \tag{41}$$

$$\frac{QK}{N^h}\left(\widehat{Q}_t + \widehat{K}_t\right) + \frac{q^B B^m}{N^h}\left(\widehat{q}_t^B + \widehat{B}_t^m\right) - \frac{D^m}{N^h}\widehat{D}_t^m = \widehat{N}_t^h \tag{42}$$

$$\widehat{R}_t^a = \frac{1}{R}\widehat{N}_t^h + \frac{\Pi^r}{RN^h}\widehat{\Pi}_t^r + \frac{1}{RN^h}\widetilde{\Pi}_t^I + \frac{\Pi^m}{RN^h}\widehat{\Pi}_t^m - \widehat{N}_{t-1}^h \tag{43}$$

Production Sector

$$\widehat{Y}_t^m = \widehat{\gamma}_t^{\text{TFP}} + \alpha\left(\widehat{U}_t + \widehat{\xi}_t + \widehat{K}_{t-1}\right) + (1-\alpha)\widehat{L}_t \tag{44}$$

$$\widehat{K}_t = \xi\left(\widehat{\xi}_t + \widehat{K}_{t-1}\right) + \frac{I_{ss}}{K}\widehat{I}_t \tag{45}$$

$$\widehat{Y}_t^m + \widehat{P}_t^m = \widehat{K}_{t-1} + \widehat{\xi}_t + (1+\nu)\widehat{U}_t \tag{46}$$

$$\widehat{W}_t = \widehat{Y}_t^m + \widehat{P}_t^m - \widehat{L}_t \tag{47}$$

$$\widehat{\delta}_t = (1+\nu)\,\widehat{U}_t \tag{48}$$

$$Q\widehat{Q}_t = \varphi_I\,(1+\beta\Lambda)\,\widehat{I}_t - \varphi_I \widehat{I}_{t-1} + \varphi_I \beta\Lambda \widehat{I}_{t+1} \tag{49}$$

$$\widehat{F}^p_t = \frac{P^m Y}{F^p}\left(\widehat{P}^m_t + \widehat{Y}_t\right) + \beta\Lambda\theta\Pi^{\sigma_p(1-\gamma_p)}\left(\widehat{\Lambda}_{t+1} + \sigma_p \widehat{\Pi}_{t+1} - \sigma_p\gamma_p\widehat{\Pi}_t + \widehat{F}^p_{t+1}\right) \tag{50}$$

$$\widehat{Z}^p_t = \frac{Y}{Z^p}\widehat{Y}_t + \beta\Lambda\theta\Pi^{(\sigma_p-1)(1-\gamma_p)}\left[\widehat{\Lambda}_{t+1} + (\sigma_p-1)\,\widehat{\Pi}_{t+1} - \gamma_p\,(\sigma_p-1)\,\widehat{\Pi}_t + \widehat{Z}^p_{t+1}\right] \tag{51}$$

$$\widehat{\Pi}_t = \theta\gamma_p\Pi^{(1-\sigma_p)(\gamma_p-1)}\widehat{\Pi}_{t-1} + (1-\theta)\,\frac{\Pi^{*(1-\sigma_p)}}{\Pi^{1-\sigma_p}}\widehat{\Pi}^*_t \tag{52}$$

$$\widehat{\Pi}^*_t = \widehat{\Pi}_t + \widehat{F}^p_t - \widehat{Z}^p_t \tag{53}$$

$$\widehat{\mu}_t = -\sigma_p\,(1-\theta)\,\frac{\Pi^{*(-\sigma_p)}}{\mu\Pi^{-\sigma_p}}\left(\widehat{\Pi}^*_t - \widehat{\Pi}_t\right) + \theta\frac{\Pi^{(-\gamma_p\sigma_p)}}{\mu\Pi^{-\sigma_p}}\left[\widehat{\mu}_{t-1} - \sigma_p\left(\gamma_p\widehat{\Pi}_{t-1} - \widehat{\Pi}_t\right)\right] \tag{54}$$

$$\widehat{Y}_t = \widehat{Y}^m_t - \widehat{\mu}_t \tag{55}$$

Central Bank

$$T\widehat{T}_t = RD^h\left(\widehat{R}_{t-1} + \widehat{D}^h_{t-1}\right) + RD^m\left(\widehat{R}_{t-1} + \widehat{D}^m_{t-1}\right) - D^m\widehat{D}^m_t - \frac{\left(1+\rho q^B\right)B^m}{\Pi}\left(\frac{\rho q^B}{1+\rho q^B}\widehat{q}^B_t + \widehat{B}^m_{t-1} - \widehat{\Pi}_t\right) \tag{56}$$

$$D^h\widehat{D}^h_t = q^B B^m\left(\widehat{q}^B_t + \widehat{B}^m_t\right) \tag{57}$$

$$\widehat{R}^i_t = \widehat{\gamma}^{\mathrm{MP}}_t \tag{58}$$

$$\widehat{B}^m_t = \widehat{\gamma}^{\mathrm{QE}}_t \tag{59}$$

$$\widehat{R}^i_t = \widehat{R}_t - \widehat{\Pi}_{t+1} \tag{60}$$

Market Clearing

$$\widetilde{\Pi}^I_t = (Q-1)\,I\widehat{I}_t + (I - I_{ss})\,Q\widehat{Q}_t \tag{61}$$

$$\widehat{\Pi}^r_t = \widehat{Y}_t - \frac{P^m\mu}{1-P^m\mu}\left(\widehat{P}^m_t + \widehat{\mu}_t\right) \tag{62}$$

$$\widetilde{\Pi}^I_t + \Pi^r\widehat{\Pi}^r_t + \Pi^m\widehat{\Pi}^m_t = 0 \tag{63}$$

$$\widehat{\Pi}^f_t = \frac{Y^m P^m}{\Pi^f}\left(\widehat{Y}^m_t + \widehat{P}^m_t\right) - \frac{LW}{\Pi^f}\left(\widehat{L}_t + \widehat{W}_t\right) - \frac{K\xi\delta}{\Pi^f}\left(\widehat{K}_{t-1} + \widehat{\xi}_t + \widehat{\delta}_t\right) \tag{64}$$

$$\widehat{\Pi}^m_t = (1-\theta^m)\left[\frac{N^h}{\Pi^m}\widehat{N}^h_{t-1} + \frac{R}{\Pi^m}\widehat{R}_{t-1} + \frac{KQ\left(R^k-R\right)}{\Pi^m}\left(\widehat{Q}_{t-1} + \widehat{K}_{t-1} + \frac{R^k}{R^k-R}\widehat{R}^k_t - \frac{R}{R^k-R}\widehat{R}_{t-1}\right) + \frac{B^m q^B\left(R^B-R\right)}{\Pi^m}\left(\widehat{q}^B_{t-1} + \widehat{B}^m_{t-1} + \frac{R^B}{R^B-R}\widehat{R}^B_t - \frac{R}{R^B-R}\widehat{R}_{t-1}\right)\right] - \frac{\Phi\phi N^h}{\Pi^m}\left(\widehat{N}^h_{t-1} + \widehat{\phi}_{t-1}\right) \tag{65}$$

$$\widehat{\Lambda}_t = -\sigma\widehat{C}_t + \sigma\widehat{C}_{t-1} \tag{66}$$

$$\widehat{\Theta}_t^T = \widehat{L}_t + \widehat{W}_t \tag{67}$$

$$\widehat{C}_t = \frac{Y}{C}\widehat{Y}_t - \frac{I}{C}\widehat{I}_t - \frac{\xi K\delta}{C}\left(\widehat{K}_{t-1} + \widehat{\xi}_t + \widehat{\delta}_t\right) \tag{68}$$

where$\widehat{I}_t = \widehat{I_{n,t} + I_{ss}}$

A.3 Derivative of Key Equations

A.3.1 Financial market

Firstly I can use equation 64, 45, 46, 47, 48 and 38 to get the return of firm equities given the assumption $\xi = 1$

$$\widehat{R}_t^k = \widehat{\xi}_t + \frac{KQ}{KQ + \Pi^f}\widehat{Q}_t - \widehat{Q}_{t-1} + \frac{\Pi^f}{KQ + \Pi^f}\widehat{\delta}_t \tag{69}$$

which I will use it later to link the investment to the leverage ratio.

Using equation 35, 36 and 40, together with $\gamma^\nu = 1$, yields$\left(1 + \frac{\zeta}{\lambda\phi\lambda^\nu}\right)\widehat{\Omega}_{t+1} + \frac{R^B}{R^B - R}\widehat{R}_{t+1}^k - \frac{R}{R^B - R}\widehat{R}_t = \frac{\zeta}{\lambda\phi\lambda^\nu}\left(\widehat{\phi}_t + \widehat{\gamma}_t^\lambda - \widehat{R}_t\right) + \widehat{\gamma}_t^\lambda$. Then the law of motion w.r.t stochastic discount factor can be found by combining equation 34, 35, 36 and 37 as $\widehat{\Omega}_t = \widehat{\Lambda}_t + \frac{\theta^m}{1 - \theta^m + \theta^m\eta}\left[\left(1 - \frac{\zeta}{(1-\lambda)\phi\lambda^\nu}\right)\left(\widehat{\Omega}_{t+1} + \widehat{R}_t\right) + \right.$ Then plug the previous equation into this law of motion will yield

$$\widehat{\phi}_t + \widehat{\gamma}_t^\lambda = \frac{1 - \theta^m + \theta^m\eta}{\theta^m}\left[\frac{\zeta}{\phi\lambda^\nu}\left(\widehat{\phi}_{t-1} + \widehat{\gamma}_{t-1}^\lambda - \widehat{R}_{t-1}\right) + \lambda\widehat{\gamma}_{t-1}^\lambda - \lambda\left(\frac{R^k}{R^k - R}\widehat{R}_t^k - \frac{R}{R^B - R}\widehat{R}_{t-1}\right) - \widehat{\Lambda}_t\right] \tag{70}$$

with the condition that in steady state $(1 - \lambda)\phi\lambda^\nu = \zeta$.

A.3.2 Illiquid asset return

Plugging equation 61, 62, 63, 65, 33 and 39 into equation 43 yields

$$\begin{aligned}\widehat{R}_t^a = &\frac{1}{R}\left[\frac{QK}{\phi N^h}\left(\widehat{Q}_t + \widehat{K}_t\right) - \widehat{\phi}_t\right] + \frac{\lambda^b q^B B^m}{\phi N^h}\left(\widehat{q}_t^B + \widehat{B}_t^m\right) + \frac{(1-\theta^m)R^B B^m q^B}{RN^h}\frac{\rho q^B}{1 + \rho q^B}\widehat{q}_t^B \\ &+ \frac{\Pi^r}{RN^h}\widehat{\Pi}_t^r + \frac{1}{RN^h}\widetilde{\Pi}_t^I + \frac{\Pi^m}{RN^h}(1 - \theta^m)\left[\frac{N^h}{\Pi^m}\widehat{N}_{t-1}^h + \frac{R}{\Pi^m}\widehat{R}_{t-1}\right. \\ &+ \frac{KQ\left(R^k - R\right)}{\Pi^m}\left(\widehat{Q}_{t-1} + \widehat{K}_{t-1} + \frac{R^k}{R^k - R}\widehat{R}_t^k - \frac{R}{R^k - R}\widehat{R}_{t-1}\right) \qquad (71) \\ &\left. + \frac{B^m q^B\left(R^B - R\right)}{\Pi^m}\left(\widehat{q}_{t-1}^B + \widehat{B}_{t-1}^m - \frac{R}{R^B - R}\widehat{R}_{t-1}\right)\right] - \frac{\Phi\phi N^h}{RN^h}\left(\widehat{N}_{t-1}^h + \widehat{\phi}_{t-1}\right) \\ &- \frac{(1-\theta^m)R^B B^m q^B}{RN^h}\left(\widehat{\Pi}_t + \widehat{q}_{t-1}^B\right) \qquad (72)\end{aligned}$$

A.3.3 General equilibrium

The redistribution effect can be identified by plugging the consumption of hand-to-mouth household 28 and 29 into the aggregate goods market clearing condition 68 and 32robustness

$$Y_C\widehat{Y}_t - I_C\widehat{I}_t - K_C\delta\left(\widehat{K}_{t-1} + \widehat{\xi}_t + \widehat{\delta}_t\right) = h_C^n\widehat{C}_t^n + \varphi_1^h\widehat{T}_t + \varphi_2^h\widehat{\Theta}_t^T$$

where $\varphi_1^h = h_C^w T_{C^w} + h_C^p T_{C^p}$ and $\varphi_2^h = h_C^w\Theta_{C^w}^T + h_C^p\Theta_{C^p}^T$

This equation can be further simplified by equation 46, 47, 48 and 67

$$\left(Y_C - \delta K_C - \varphi_2^h\right)\widehat{Y}_t - I\hat{I}_t - \left(K_C\delta + \varphi_2^h\right)\left(\widehat{P}_t^m + \widehat{\mu}_t\right) = h_C^n\widehat{C}_t^n + \varphi_1^h\widehat{T}_t$$

Further I use the budget constraint of the non hand-to-mouth household $C^n\widehat{C}_t^n - \frac{D^h}{h^n}\widehat{D}_t^h = \frac{(1-\tau)WL}{h^n}\left(\widehat{W}_t + \widehat{L}_t\right) - \frac{RD^h}{h^n}\left(\widehat{R}_{t-1} + \widehat{D}_{t-1}^h\right) + T\widehat{T}_t$ and long-term bonds clearing condition 57 to get

$$\left(Y_C - \delta K_C - \varphi_2^h + \frac{\varphi_1^h\left(1-\tau\right)WL}{h^nT}\right)\widehat{Y}_t = I\widehat{I}_t + \left(K_C\delta + \varphi_2^h - \frac{\varphi_1^h}{h^nT}\left(1-\tau\right)WL\right)\left(\widehat{P}_t^m + \widehat{\mu}_t\right) + \left(h_C^n + \varphi_3^h\right)\widehat{C}_t^n + \frac{\varphi_1^h}{h^nT}\left[-q^BB^m\left(\widehat{q}_t^B + \widehat{B}_t^m\right) + RD^h\widehat{R}_{t-1} + Rq^BB^m\left(\widehat{q}_{t-1}^B + \widehat{B}_{t-1}^m\right)\right] \quad (73)$$

where $\varphi_3^h = h_C^w\frac{C^n}{C^w} + h_C^p\frac{C^n}{C^p}$

Furthermore I can combine equation 31, 44, 46 and 47 to get the consumption of non hand-to-mouth household is

$$\widehat{C}_t^n = -\frac{\psi}{\sigma}\widehat{Y}_t + \left(\frac{1}{\sigma} + \frac{(1+\psi)\alpha}{(1-\alpha)\sigma}\right)\widehat{P}_t^m - \frac{\psi}{\sigma}\widehat{\mu}_t + \frac{1+\psi}{(1-\alpha)\sigma}\widehat{\gamma}_t^{\text{TFP}} - \frac{(1+\psi)\alpha\nu}{(1-\alpha)\sigma}\widehat{U}_t \quad (74)$$

Then plug above equation back into equation 73

$$\left(Y_C - \delta K_C - \varphi_2^h + \frac{\varphi_1^h\left(1-\tau\right)WL}{h^nT} + \left(h_C^n + \varphi_3^h\right)\frac{\psi}{\sigma}\right)\widehat{Y}_t = I\widehat{I}_t + \left(K_C\delta + \varphi_2^h - \frac{\varphi_1^h}{h^nT}\left(1-\tau\right)WL\right)\left(\widehat{P}_t^m + \widehat{\mu}_t\right) + \frac{\varphi_1^h}{h^nT}\left[-q^BB^m\left(\widehat{q}_t^B + \widehat{B}_t^m\right) + RD^h\widehat{R}_{t-1} + Rq^BB^m\left(\widehat{q}_{t-1}^B + \widehat{B}_{t-1}^m\right)\right] + \left(h_C^n + \varphi_3^h\right)\left[\left(\frac{1}{\sigma} + \frac{(1+\psi)\alpha}{(1-\alpha)\sigma}\right)\widehat{P}_t^m - \frac{\psi}{\sigma}\widehat{\mu}_t + \frac{1+\psi}{(1-\alpha)\sigma}\widehat{\gamma}_t^{\text{TFP}} - \frac{(1+\psi)\alpha\nu}{(1-\alpha)\sigma}\widehat{U}_t\right] \quad (75)$$

If I assume that the investment is a constant fraction of output and fixed price, then there will exist $\widehat{R}_{t-1} = \widehat{P}_t^m = \widehat{\mu}_t = 0$. From equation 75 you can see the coefficient $\frac{Y_C - \delta K_C - \varphi_2^h + \frac{\varphi_1^h(1-\tau)WL}{h^nT} + \left(h_C^n + \varphi_3^h\right)\frac{\psi}{\sigma}}{\frac{\varphi_1^h}{h^nT}q^BB^m}$ is just the redistribution effect of the unconventional monetary policy. The hand-to-mouth household generate a multiplier of the monetary policy through additionally terms $-\varphi_2^h + \frac{\varphi_1^h(1-\tau)WL}{h^nT} + \left(h_C^n + \varphi_3^h\right)\frac{\psi}{\sigma}$ and $\frac{\varphi_1^h}{h^nT}$ which is firstly mentioned by

Bilbiie (2020).

By the same logic I can also combine the non hand-to-mouth household's budget constraint and central bank's budget constraint 56 and 57 which yields

$$\begin{aligned} C^n\widehat{C}^n_t - \frac{q^B B^m}{h^n}\left(\widehat{q}^B_t + \widehat{B}^m_t\right) &= \frac{(1-\tau)WL}{h^n}\left(\widehat{Y}_t + \widehat{P}^m_t + \widehat{\mu}_t\right) - \left(\frac{1}{h^n}-1\right) RD^h\widehat{R}_{t-1} \\ &- \left(\frac{1}{h^n}-1\right) Rq^B B^m\left(\widehat{q}^B_{t-1} + \widehat{B}^m_{t-1}\right) + RD^m\left(\widehat{R}_{t-1} + \widehat{D}^m_{t-1}\right) \\ &- D^m\widehat{D}^m_t - \left(1+\rho q^B\right) B^m\left(\frac{\rho q^B}{1+\rho q^B}\widehat{q}^B_t + \widehat{B}^m_{t-1} - \widehat{\Pi}_t\right) \end{aligned}$$

To eliminate the money that financial institutions borrow from t he central bank $\widehat{D}^m_t$ I plug equation 33 and 42 into above equation to get

$$\begin{aligned} C^n\widehat{C}^n_t =& \frac{(1-\tau)WL}{h^n}\left(\widehat{Y}_t + \widehat{P}^m_t + \widehat{\mu}_t\right) - \left(\frac{1}{h^n}-1\right) RD^h\widehat{R}_{t-1} \\ &- \left(\frac{1}{h^n}-1\right) Rq^B B^m\left(\widehat{q}^B_{t-1} + \widehat{B}^m_{t-1}\right) + RD^m\left(\widehat{R}_{t-1} + \widehat{D}^m_{t-1}\right) - \left(1-\frac{1}{\phi}\right) QK\left(\widehat{Q}_t + \widehat{K}_t\right) \\ &- N^h\widehat{\phi}_t + \left(\frac{1}{h^n} - 1 + \frac{\lambda^b}{\phi}\right) q^B B^m\left(\widehat{q}^B_t + \widehat{B}^m_t\right) - \left(1+\rho q^B\right) B^m\left(\frac{\rho q^B}{1+\rho q^B}\widehat{q}^B_t + \widehat{B}^m_{t-1} - \widehat{\Pi}_t\right) \end{aligned}$$

Plugging equation 70 and 75 into equation above equation yields

$$\begin{aligned} -C^n\frac{\psi}{\sigma}\widehat{Y}_t =& -C^n\left(\frac{1}{\sigma} + \frac{(1+\psi)\alpha}{(1-\alpha)\sigma}\right)\widehat{P}^m_t + C^n\frac{\psi}{\sigma}\widehat{\mu}_t - C^n\frac{1+\psi}{(1-\alpha)\sigma}\widehat{\gamma}^{\text{TFP}}_t + C^n\frac{(1+\psi)\alpha\nu}{(1-\alpha)\sigma}\widehat{U}_t \\ &+ \frac{(1-\tau)WL}{h^n}\left(\widehat{Y}_t + \widehat{P}^m_t + \widehat{\mu}_t\right) - \left(\frac{1}{h^n}-1\right) RD^h\widehat{R}_{t-1} \\ &- \left(\frac{1}{h^n}-1\right) Rq^B B^m\left(\widehat{q}^B_{t-1} + \widehat{B}^m_{t-1}\right) + RD^m\left(\widehat{R}_{t-1} + \widehat{D}^m_{t-1}\right) - \left(1-\frac{1}{\phi}\right) QK\left(\widehat{Q}_t + \widehat{K}_t\right) \\ &+ N^h\frac{1-\theta^m+\theta^m\eta}{\theta^m}\lambda\frac{R^B}{R^B-R}\widehat{R}^k_t + N^h\widehat{\gamma}^\lambda_t \\ &- N^h\frac{1-\theta^m+\theta^m\eta}{\theta^m}\left[\frac{\zeta}{\phi\lambda^\nu}\left(\widehat{\phi}_{t-1} + \widehat{\gamma}^\lambda_{t-1} - \widehat{R}_{t-1}\right) + \lambda\widehat{\gamma}^\lambda_{t-1} + \lambda\frac{R}{R^B-R}\widehat{R}_{t-1} - \widehat{\Lambda}_t\right] \\ &+ \left(\frac{1}{h^n} - 1 + \frac{\lambda^b}{\phi}\right) q^B B^m\left(\widehat{q}^B_t + \widehat{B}^m_t\right) - \left(1+\rho q^B\right) B^m\left(\frac{\rho q^B}{1+\rho q^B}\widehat{q}^B_t + \widehat{B}^m_{t-1} - \widehat{\Pi}_t\right) \end{aligned}$$

This equation can set the relationship between unconventional monetary policy and real output by combining equation 45, 49, 69 and 73

$$
\begin{aligned}
0 = & -C^n\left(\frac{1}{\sigma}+\frac{(1+\psi)\alpha}{(1-\alpha)\sigma}\right)\widehat{P}_t^m + C^n\frac{\psi}{\sigma}\widehat{\mu}_t - C^n\frac{1+\psi}{(1-\alpha)\sigma}\widehat{\gamma}_t^{\text{TFP}} + C^n\frac{(1+\psi)\alpha\nu}{(1-\alpha)\sigma}\widehat{U}_t \\
& + \left(C^n\frac{\psi}{\sigma}+\frac{(1-\tau)WL}{h^n}+\varphi_1^m\varphi_4^h\right)\widehat{Y}_t + \frac{(1-\tau)WL}{h^n}\left(+\widehat{P}_t^m+\widehat{\mu}_t\right) - \left(\frac{1}{h^n}-1\right)RD^h\widehat{R}_{t-1} \\
& - \left(\frac{1}{h^n}-1\right)Rq^BB^m\left(\widehat{q}_{t-1}^B+\widehat{B}_{t-1}^m\right) + RD^m\left(\widehat{R}_{t-1}+\widehat{D}_{t-1}^m\right) - \left(1-\frac{1}{\phi}\right)QK\left(\widehat{\xi}_t+\widehat{K}_{t-1}\right) \\
& - \left(N^h\frac{1-\theta^m+\theta^m\eta}{\theta^m}\lambda\frac{R^k}{R^k-R}\frac{1}{KQ+\Pi^f}-1+\frac{1}{\phi}\right)K\varphi_I(1-\beta\Lambda)\widehat{I}_{t-1} \\
& - \varphi_1^m\left[\left(K_C\delta+\varphi_2^h-\frac{\varphi_1^h}{h^nT}(1-\tau)WL\right)\left(\widehat{P}_t^m+\widehat{\mu}_t\right)+\frac{\varphi_1^h}{h^nT}\left[RD^h\widehat{R}_{t-1}+Rq^BB^m\left(\widehat{q}_{t-1}^B+\widehat{B}_{t-1}^m\right)\right]\right. \\
& \left. + \left(h_C^n+\varphi_3^h\right)\left[\left(\frac{1}{\sigma}+\frac{(1+\psi)\alpha}{(1-\alpha)\sigma}\right)\widehat{P}_t^m-\frac{\psi}{\sigma}\widehat{\mu}_t+\frac{1+\psi}{(1-\alpha)\sigma}\widehat{\gamma}_t^{\text{TFP}}-\frac{(1+\psi)\alpha\nu}{(1-\alpha)\sigma}\widehat{U}_t\right]\right] \\
& + N^h\frac{1-\theta^m+\theta^m\eta}{\theta^m}\lambda\frac{R^k}{R^k-R}\left(\widehat{\xi}_t-\widehat{Q}_{t-1}+\frac{\Pi^f}{KQ+\Pi^f}\widehat{\delta}_t\right)+N^h\widehat{\gamma}_t^\lambda \\
& - N^h\frac{1-\theta^m+\theta^m\eta}{\theta^m}\left[\frac{\zeta}{\phi\lambda^\nu}\left(\widehat{\phi}_{t-1}+\widehat{\gamma}_{t-1}^\lambda-\widehat{R}_{t-1}\right)+\lambda\widehat{\gamma}_{t-1}^\lambda+\lambda\frac{R}{R^B-R}\widehat{R}_{t-1}-\widehat{\Lambda}_t\right] \\
& + \left(\frac{1}{h^n}-1+\frac{\lambda^b}{\phi}+\varphi_1^m\frac{\varphi_1^h}{Th^n}\right)q^BB^m\left(\widehat{q}_t^B+\widehat{B}_t^m\right) - \left(1+\rho q^B\right)B^m\left(\frac{\rho q^B}{1+\rho q^B}\widehat{q}_t^B+\widehat{B}_{t-1}^m-\widehat{\Pi}_t\right)
\end{aligned}
$$

where $\varphi_1^m = \left(N^h\frac{1-\theta^m+\theta^m\eta}{\theta^m}\lambda\frac{R^k}{R^k-R}\frac{1}{KQ+\Pi^f}-1+\frac{1}{\phi}\right)\frac{\varphi_I}{\delta}-\left(1-\frac{1}{\phi}\right)Q$ and $\varphi_4^h = Y_C-\delta K_C-\varphi_2^h+\frac{\varphi_1^h(1-\tau)WL}{h^nT}+\left(h_C^n+\varphi_3^h\right)\frac{\psi}{\sigma}$.

When the price and depreciation are fixed I will get the unconventional monetary policy effect to the output as $-\frac{\frac{1}{h^n}-1+\frac{\lambda^b}{\phi}+\varphi_1^m\frac{\varphi_1^h}{Th^n}}{C^n\frac{\psi}{\sigma}+\frac{(1-\tau)WL}{h^n}+\varphi_1^m\varphi_4^h}q^BB^m$ and $\frac{\rho}{C^n\frac{\psi}{\sigma}+\frac{(1-\tau)WL}{h^n}+\varphi_1^m\varphi_4^h}q^BB^m$.

A.4 Calibration for production sector

All the parameters related to production sector are selected from literature. The possibility of retailer to retain price is 0.85 which is a balance within 0.75 of Sims et al. (2022), 0.779 of Gertler and Karadi (2011) and 0.92 of Karadi and Nakov (2021). The elasticity of substitution and capital production cost are directly borrowed from Gertler and Karadi (2011). The utilization elasticity of marginal depreciation rate is estimated by Christiano et al. (2014).

Table 3: Parameter Values

Parameter	Value	Description
θ	0.85	Possibility to retain price
σ_p	4.167	Inverse elasiticity of substitution between different intermediate goods
γ_p	0	Parameter for retained price that grow as aggregate inflation
α	0.36	Capital share in production function

Table 3 – Continued

Parameter	Value	Description
ν	2.54	utilization elasticity of marginal depreciation rate
$\bar{\delta}$	0.025	Depreciation rate in ss
ψ_I	1.728	Capital production cost
$\bar{\xi}$	1.000	Capital effectiveness in ss
θ_r^{QE}	0.92	AR1 coefficient of QE policy rule
$\bar{u}$	1.000	utilizebar

B Extensions

B.1 Endogenous illiquid asset withdrawing

In baseline model the illquid asset withdrawing X_t^{nHtM} and X_t^{wHtM} are exogenous and fixed at steady state for conventional purpose as it shuts down the substitution effect between liquid and illiquid asset in liquidity and interest rate channel. To illustrate that the baseline model is good enough to unveil the two channels of unconventional monetary policy, I change the exogenous illiquid asset withdrawing to endogenous following the extension that Cui and Sterk (2021) did. Either wealthy hand-to-mouth household or non-hand-to-mouth household now will select the their optimal illiquid asset withdrawing X_t^i because of an adjustment friction in utility base. Therefore the marginal cost of altering the portfolio of investment should be equal to the marginal benefit (which is the marginal consumption in this setting) as equation 76 shows.

$$\gamma_1 + \frac{X_t^{\gamma_3}}{\gamma_2^{1+\gamma_3}} = C_t^{-\sigma} \tag{76}$$

The adjustment cost function is standard and proposed by Kaplan et al. (2018) where γ_1 is the extend of friction in linear part and γ_3 governs the curvature of the cost function which in fact is the nonlinear part. γ_2 works as a resealing factor. Following Cui and Sterk (2021) I set $\gamma_2 = 1.5$ and $\gamma_3 = 5$ which are not unique to the result and other value will also work. Then the linear part γ_1, which is the least important and does not display in linearization system, is pinned down by matching the same steady state of consumption and illquid asset withdrawing in baseline model.

Figure 6 shows that the endogeneity of illiquid asset withdrawing amplifies the power of unconventional monetary policy yet the magnitude is not large. In addition to that all the patterns and directions between the exogenous and endogenous model are same which demonstrates that the baseline model works well at uncovering the two channels.

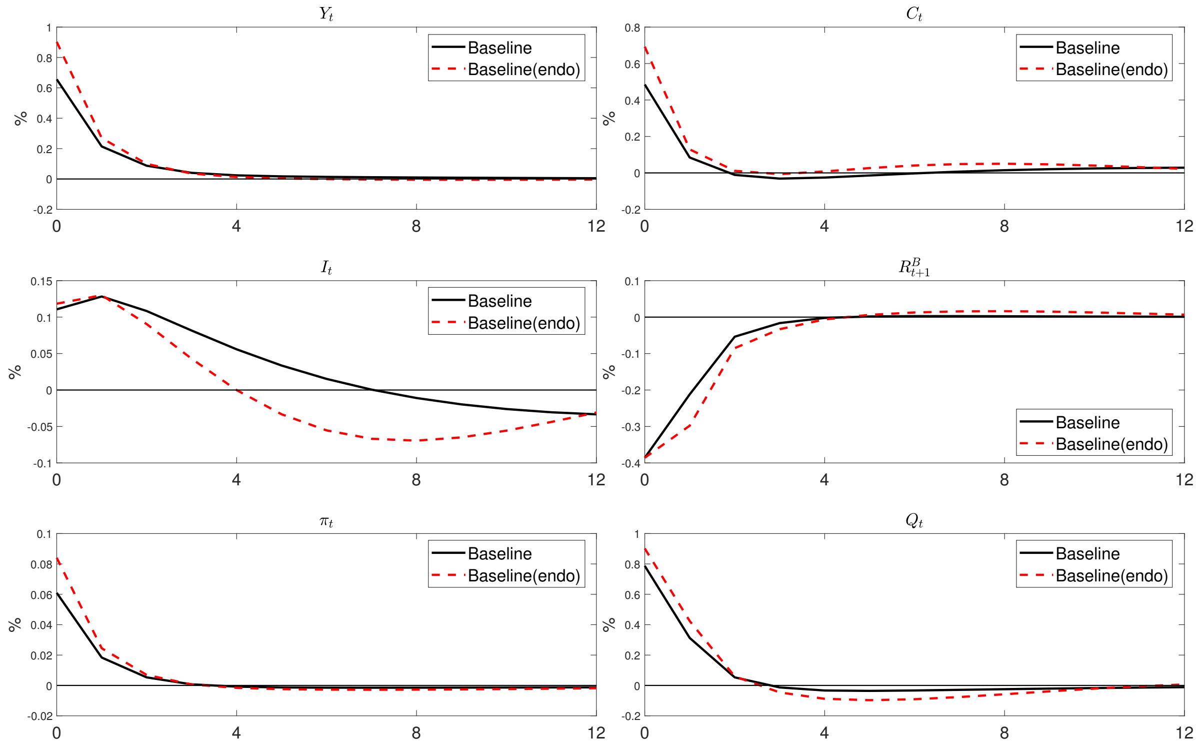


Figure 6: Exogenous vs Endogenous Illiquid asset withdrawing

B.2 Model vs Empirical result

C Data

C.1 Data used in VAR estimation

C.2 Data used in Model Calibration

D Robustness check

D.1 Off-diagonal zero restriction

Since I want to identify the liquidity channel and interest rate channel of monetary policy, I can write $\boldsymbol{\varepsilon}'_{1t}$ as $\boldsymbol{\varepsilon}'_{1t} = \left[\begin{array}{cc} \varepsilon^s_{1t} & \varepsilon^d_{1t} \end{array} \right]'$ where ε^s_{1t} and ε^d_{1t} denote the liquidity shock and demand shock.

I made a restriction on $\boldsymbol{\Phi}$ and assume it is a diagonal matrix such that

$$\boldsymbol{\Phi} = \left[\begin{array}{cc} \alpha^s & 0 \\ 0 & \alpha^d \end{array} \right] \tag{77}$$

The meaning of this restriction is that the instrument variables do not have cross effect on different shock.

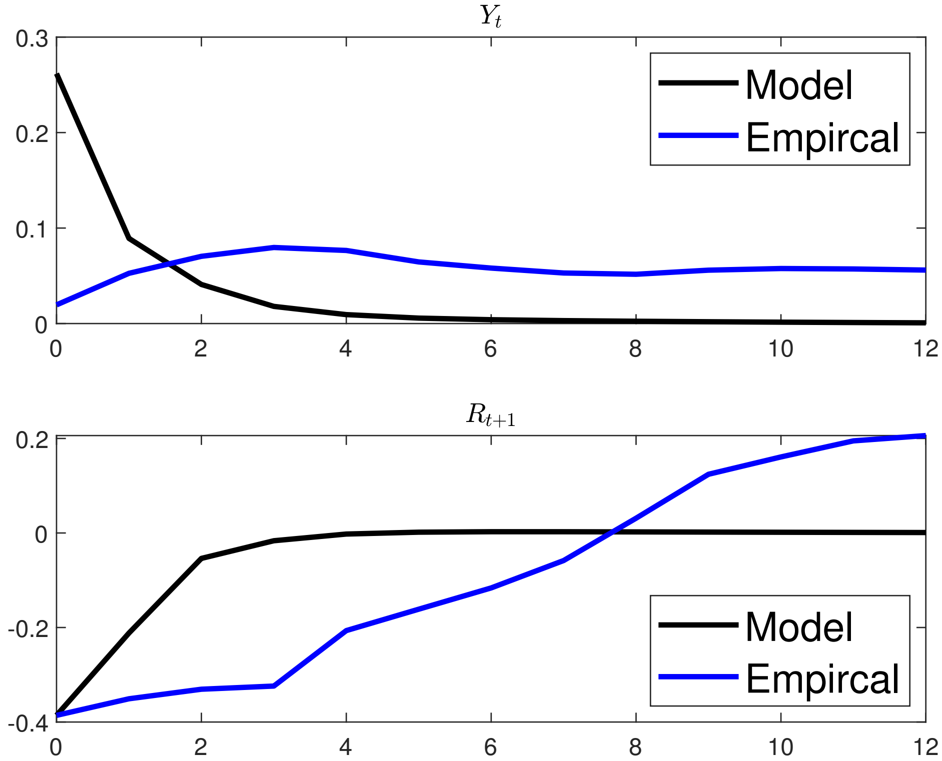


Figure 7

For

$$\mathbf{m}_t = \left[\begin{array}{cc} m_t^s & m_t^d \end{array} \right]'$$

we would have the assumption $\text{cov}\left(m_t^s, \varepsilon_{1t}^d\right) = 0$ and $\text{cov}\left(m_t^d, \varepsilon_{1t}^s\right) = 0$. This assumption can be true as long as I find appropriate instrument variables. Similar to the high frequency identification literature, the instrument helps to identify the effect of liquidity channel of monetary policy is the change of future contract price during the treasury bonds issuing announcement day. The instrument helps to identify the effect of interest rate channel of monetary policy is the change of future contract price during the FOMC announcement day.

Write the covariance matrix $\boldsymbol{\Sigma}_{\boldsymbol{mu}'}$ into partition

$$\boldsymbol{\Sigma}_{\boldsymbol{mu}'} = \left[\begin{array}{cc} \Sigma_{mu'_{11}} & \boldsymbol{\Sigma}_{\boldsymbol{mu}'_{12}} \\ \Sigma_{mu'_{21}} & \boldsymbol{\Sigma}_{\boldsymbol{mu}'_{22}} \end{array} \right]$$

and plug into assumption 77 to get

$$\boldsymbol{\beta}_{21} = \left(\Sigma_{mu'_{11}}^{-1} \boldsymbol{\Sigma}_{\boldsymbol{mu}'_{21}}\right) \beta_{11}$$

and

$$\boldsymbol{\beta}_{22} = \left(\Sigma_{mu'_{12}}^{-1} \boldsymbol{\Sigma}_{\boldsymbol{mu}'_{22}}\right) \beta_{12}$$

because $\boldsymbol{\Phi}$ is a diagonal matrix and I can easily subtract α^s and α^d in each row of $\boldsymbol{\Phi}\boldsymbol{\beta}_1'$ out separately trough each row of $\boldsymbol{\Sigma}_{\boldsymbol{mu}'}$

Use the method proposed by Gertler and Karadi (2015) I can easily identify $\boldsymbol{\beta}_{21}$ and β_{11}. Additionally it is easy to prove that for $\boldsymbol{\beta}_{22}$ and β_{12}, $\boldsymbol{\beta}_{22}\beta_{12}^{-1}$ can be estimated by estimating $\left(\Sigma_{mu'_{21}}^{-1}\boldsymbol{\Sigma}_{\boldsymbol{mu}'_{22}}\right)$. Then we can solve β_{12} up to a sign convention that[49]

$$\beta_{12}^2 = b_{12}^2 = \Sigma_{11} - \left(b_{11}^2 + \boldsymbol{b}_{13}\boldsymbol{b}'_{13}\right) \tag{78}$$

where

$$\boldsymbol{b}_{13}\boldsymbol{b}'_{13} = \left(\boldsymbol{\Sigma}_{31} - \frac{\boldsymbol{b}_{32}}{b_{12}}\Sigma_{11} - \boldsymbol{b}_{31}b_{11} + \frac{\boldsymbol{b}_{32}}{b_{12}}b_{11}^2\right)' D^{-1}\left(\boldsymbol{\Sigma}_{31} - \frac{\boldsymbol{b}_{32}}{b_{12}}\Sigma_{11} - \boldsymbol{b}_{31}b_{11} + \frac{\boldsymbol{b}_{32}}{b_{12}}b_{11}^2\right)$$

and

$$\begin{aligned} D = \boldsymbol{\Sigma}_{33} + \frac{\boldsymbol{b}_{32}}{b_{12}}\Sigma_{11}\frac{\boldsymbol{b}'_{32}}{b_{12}} - \boldsymbol{b}_{31}\boldsymbol{b}'_{31} - \frac{\boldsymbol{b}_{32}}{b_{12}}b_{11}^2\frac{\boldsymbol{b}'_{32}}{b_{12}} - \\ \left(\frac{\boldsymbol{b}_{32}}{b_{12}}\boldsymbol{\Sigma}'_{31} + \boldsymbol{\Sigma}_{31}\frac{\boldsymbol{b}'_{32}}{b_{12}} - \frac{\boldsymbol{b}_{32}}{b_{12}}\boldsymbol{b}'_{31}b_{11} - \boldsymbol{b}_{31}b_{11}\frac{\boldsymbol{b}'_{32}}{b_{12}}\right) \end{aligned}$$

The estimation result is shown below.

[49] Because of space limit, I degraded related proof process to appendix.

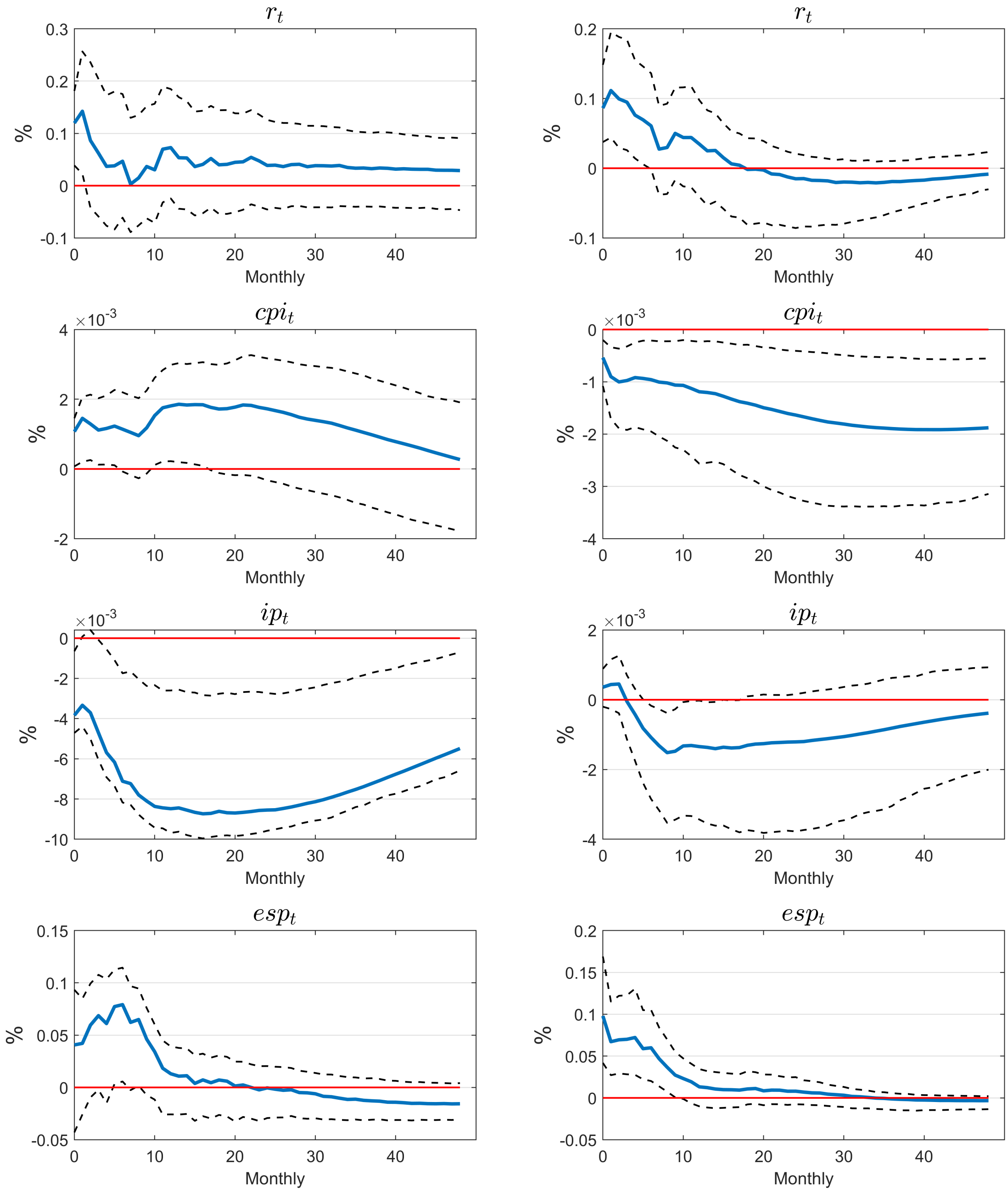


Figure 8: Liquidity and interest rate channel

The left column is related to liquidity channel of QE. The monetary authority stimulates the economy through changing the long-term bonds rate via providing liquidity and buying long-term bonds. The right column is related to interest rate channel of QE. The monetary authority stimulates the economy through changing the long-term bonds rate(people's expectation about future interest rate) directly via announcement but does not provide any liquidity to the market.

D.2 Lower triangle restriction

Alternatively I can only impose the triangle restriction on matrix $\mathbf{\Phi}$[50] such that the matrix $\mathbf{\Phi}$ is in the form

$$\mathbf{\Phi} = \begin{bmatrix} \alpha^s & 0 \\ \alpha^p & \alpha^d \end{bmatrix}$$

Similar to use the restriction 77, I can estimate b_{11}, b_{21} and $\boldsymbol{b}_{31}$ by ruling out α^s using instrument m_t^s. While it is a little bit trivial to estimate b_{12}, b_{22} and $\boldsymbol{b}_{32}$ and I provide the detailed process in another appendix section.

[50]It is not freely to impose this restriction relative to the zero on off-diagonal element as I add one more unknown. This cause the sign b_{22} undetermined. Therefore I put this restriction on the robustness check.

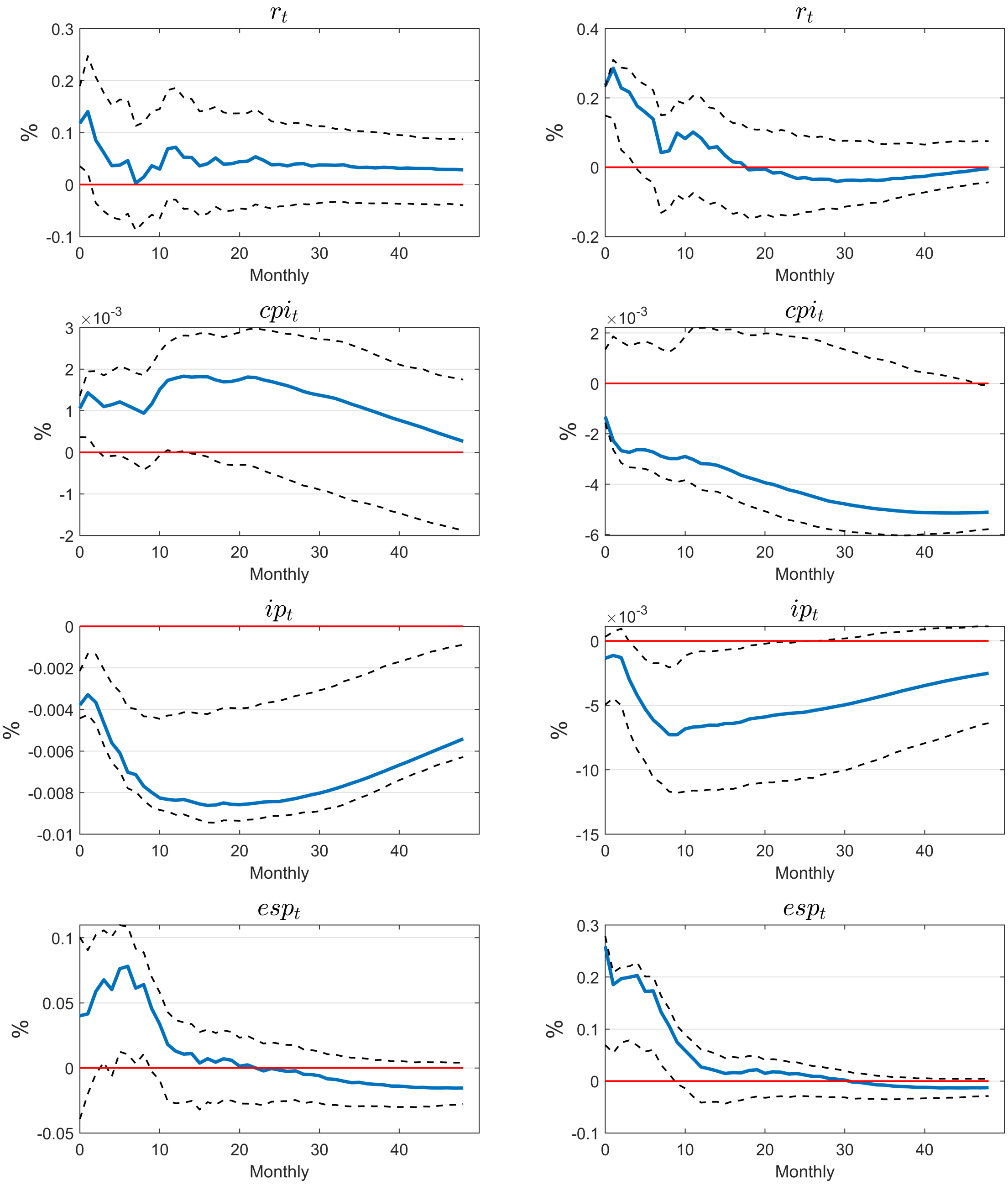


Figure 9: Liquidity and interest rate channel

D.3 Uniform prior

Alternative I can identify β_1 through inequality restriction and bayesian estimation. Because of equation 24, when I only want to identify β_1 and do not care about β_2, it could be fully identified as long as Φ is know. Because of the constraint of freedom, when $k = 2$ only half of Φ

can be identified as long as I do not want impose any restriction more on $\boldsymbol{\beta}_1$. Inspired by the half-information identification problem, I can use the standard and canonical method that are used to identify the $\boldsymbol{B}$ matrix[51] which is already widely developed. Firstly it is easy and valuable to notice that

$$\boldsymbol{\Phi}\boldsymbol{\Phi}' = \boldsymbol{\Sigma}_{\boldsymbol{mu}_1'} \left(\boldsymbol{\Sigma}_{\boldsymbol{mu}_1'}'^{-1} \boldsymbol{s}_{11} \boldsymbol{s}_{11}' \right)^{-1}$$

Then taking the Cholesky decomposition on $\boldsymbol{\Phi}\boldsymbol{\Phi}'$ yields the lower triangle matrix $\boldsymbol{\Phi}_{tr}$. $\boldsymbol{\Phi}$ will be identified up to the rotation matrix Q such that $\boldsymbol{\Phi} = \boldsymbol{\Phi}_{tr} Q$.

Therefore I can impose inequality constraint to identify $\boldsymbol{\Phi}$(then the $\boldsymbol{\beta}_1$ is identified through equation 24), similar to the sign restriction proposed by Uhlig (2005). Here I use the method proposed by Rubio-Ramirez et al. (2010) with uniform prior of Q to identify[52].

Draw 1000 times.

- $\kappa_1 = \kappa_2 = 0$

[51] Since in general we only have half information of matrix B which is the covariance matrix BB'.

[52] Baumeister and Hamilton (2015) argued that using the uniform prior distribution will result in a Cauchy distribution.

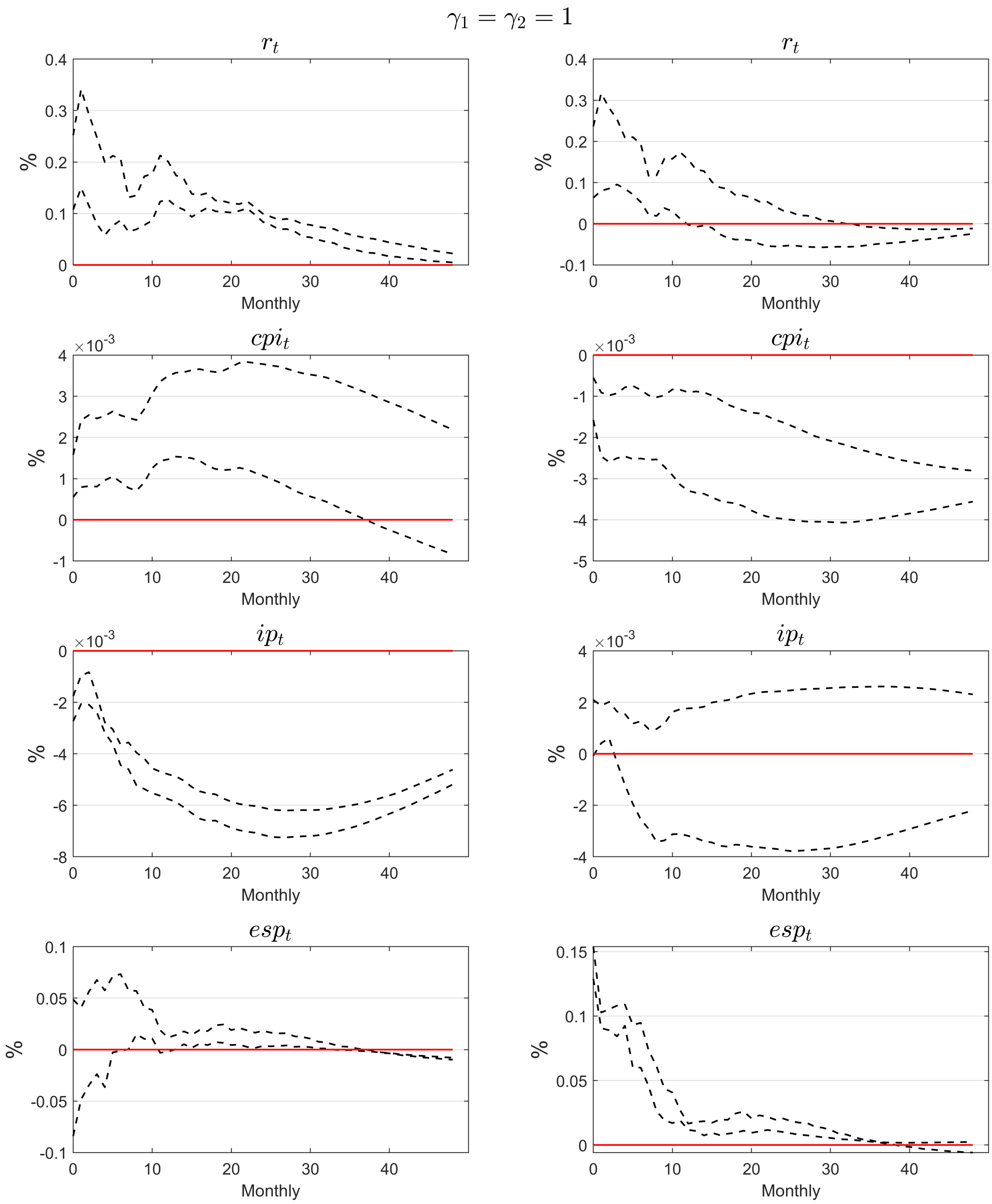


Figure 10: Liquidity and interest rate channel

- $\kappa_1 = 1$ but $\kappa_2 = 0.9$

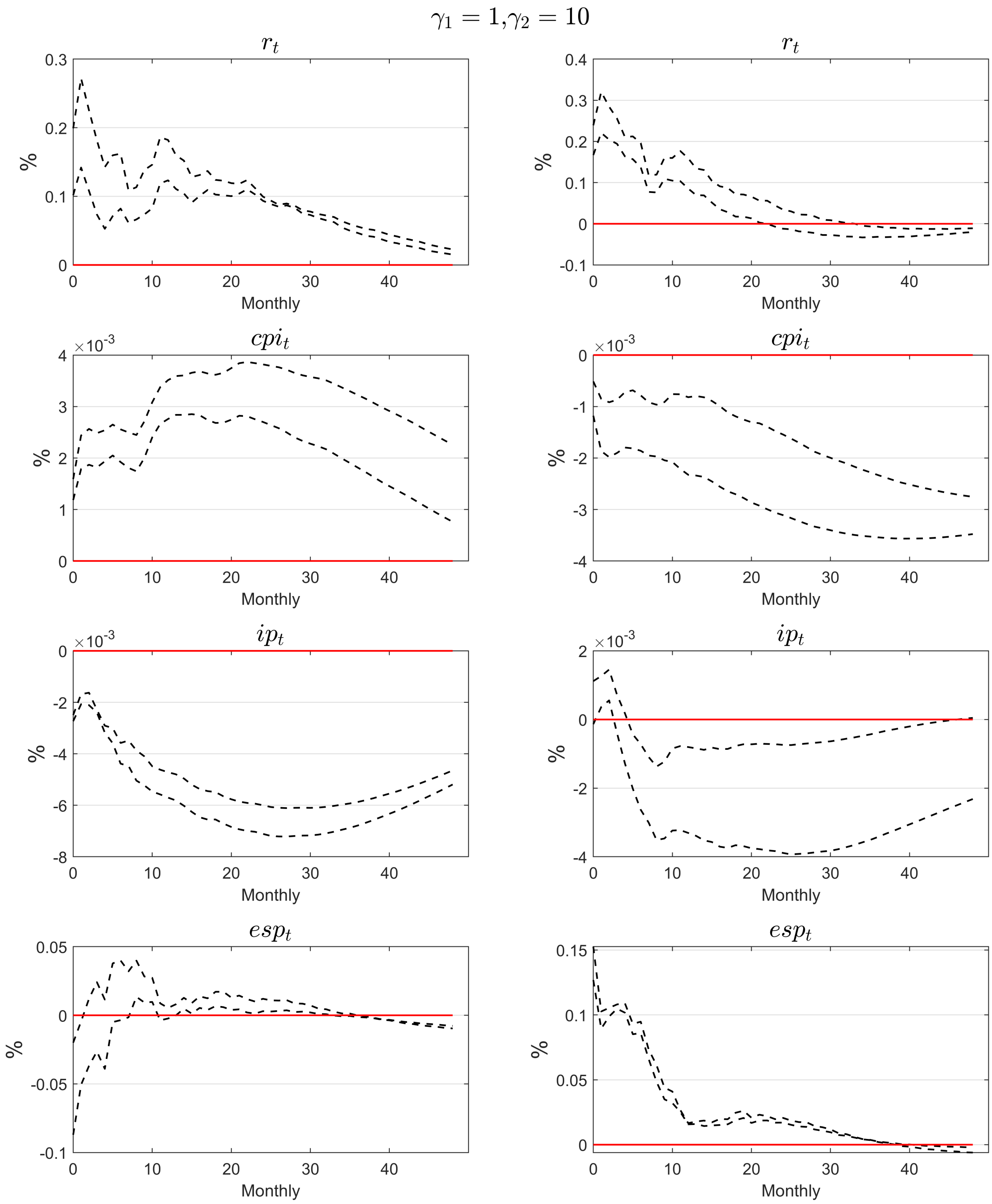


Figure 11: Liquidity and interest rate channel

D.4 Bayesian estimation using different drawing algorithm

This section presents the Bayesian estimation of VAR under more conservative algorithm 2 and 3.

Draw 1000 times with 7000 burn-in try. 90% confidence band.

Algorithm 2:

- $\kappa_1 = \kappa_2 = 0$

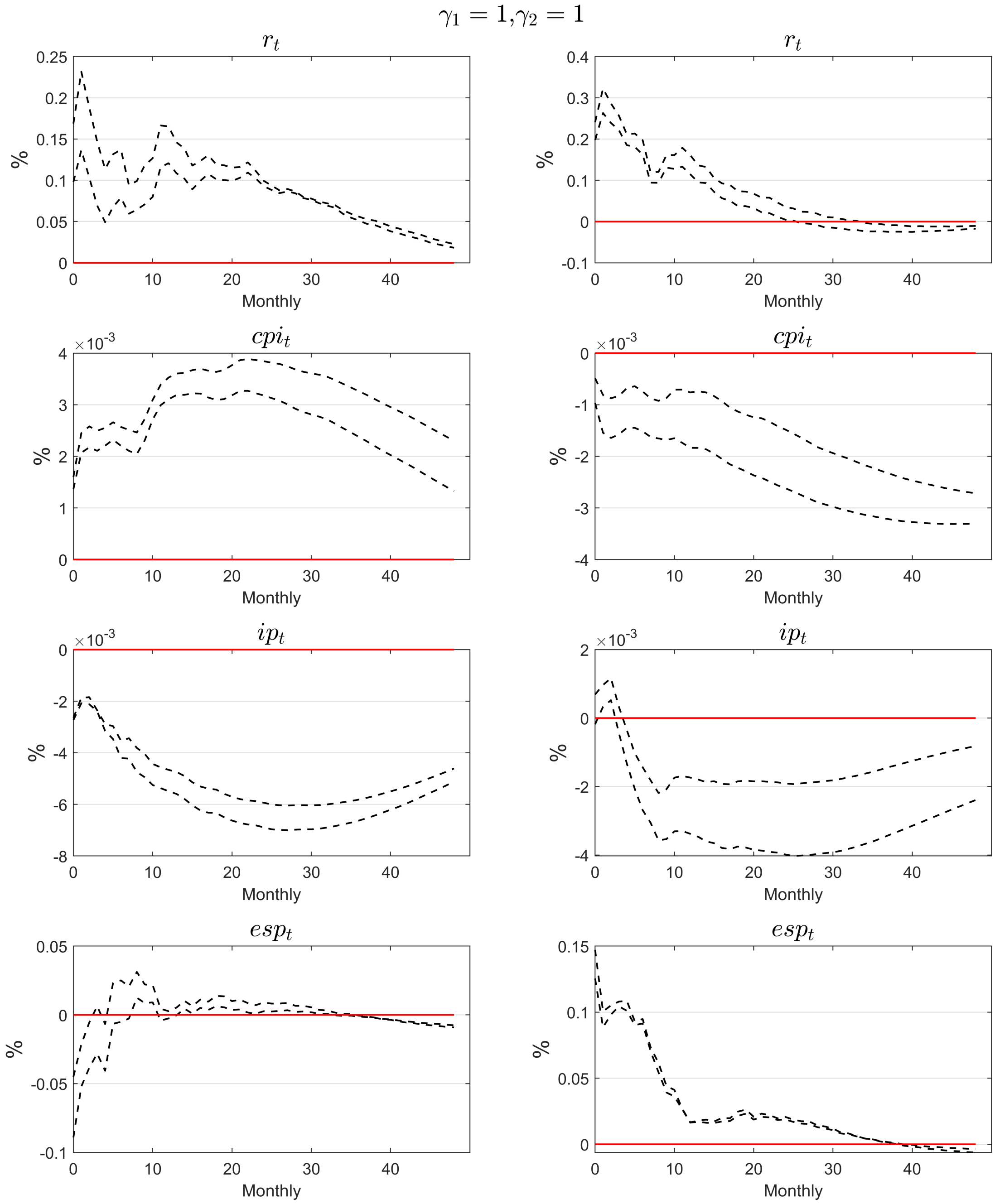


Figure 12: Liquidity and interest rate channel

Algorithm 3:

- $\kappa_1 = \kappa_2 = 0$

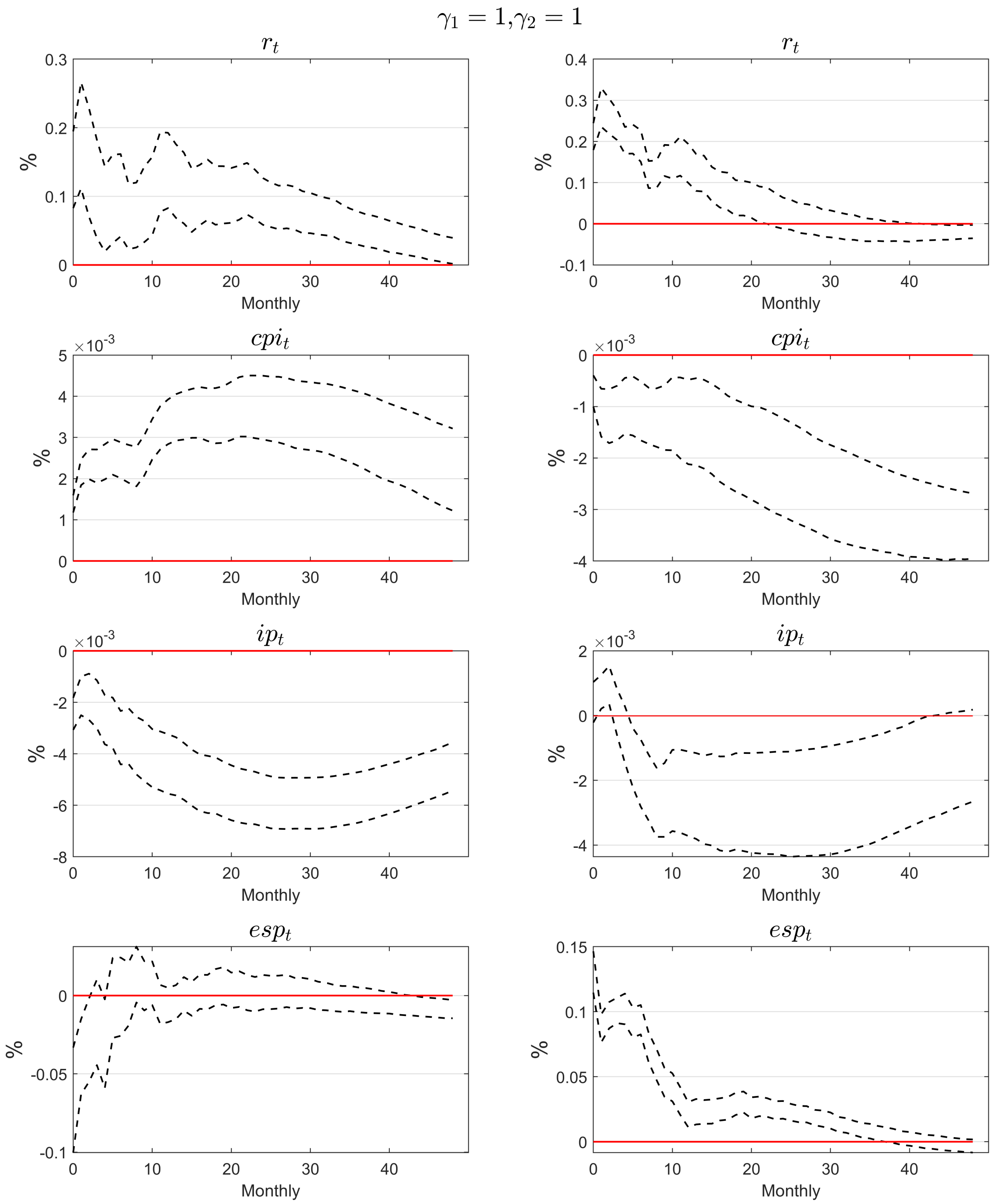


Figure 13: Liquidity and interest rate channel

E Derivative process related to IV-VAR

E.1 Estimation of coefficient under off-diagonal zero restriction

To the coefficient parameters β_{11} and β_{12} they can be directly estimated by the method used in literature because I assume the coefficient matrix of instrument variables is off-diagonal zero. Therefore the first instrument will only be correlated with the first shock, a scenario where Gertler and Karadi (2015) considered. Therefore β_{11} can be estimated up to sign convention such that

$$\beta_{11}^2 = \Sigma_{11} - \beta_{12}\beta_{12}'$$

where

$$\beta_{12}\beta_{12}' = \left(\Sigma_{21} - \frac{\beta_{21}}{\beta_{11}}\Sigma_{11}\right)' \mathbf{Q}^{-1} \left(\Sigma_{21} - \frac{\beta_{21}}{\beta_{11}}\Sigma_{11}\right)$$

and

$$\mathbf{Q} = \frac{\beta_{21}}{\beta_{11}}\Sigma_{11}\frac{\beta_{21}'}{\beta_{11}} - \left(\Sigma_{21}\frac{\beta_{21}'}{\beta_{11}} + \frac{\beta_{21}}{\beta_{11}}\Sigma_{21}'\right) + \Sigma_{22}.$$

and $\widehat{\frac{\beta_{21}}{\beta_{11}}}$ comes from the coefficient of IV regression.

Similarly another instrument can be used to estimate β_{12} and β_{22} though the process becomes a little bit more complicated.

Firstly note that because $E\left(\boldsymbol{u}_t\boldsymbol{u}_t'\right) = E\left(\boldsymbol{B}\varepsilon_t\varepsilon_t'\boldsymbol{B}'\right) = \boldsymbol{B}E\left(\varepsilon_t\varepsilon_t'\right)\boldsymbol{B}' = \boldsymbol{B}\boldsymbol{B}' = \boldsymbol{\Sigma}$ and $\boldsymbol{\beta}_{22}\beta_{12}^{-1}$ can be easily estimated, we can know that as long as $\boldsymbol{b}_{13}\boldsymbol{b}_{13}'$ is revealed, $\boldsymbol{\beta}_{22}$ and β_{12} will be uncovered from equation 78. The steps below I show how to estimate $\boldsymbol{b}_{13}\boldsymbol{b}_{13}'$.

Now I construct a matrix such that

$$\boldsymbol{V} = \boldsymbol{b}_{33} - \frac{\boldsymbol{b}_{32}\boldsymbol{b}_{13}}{b_{11}}$$

It is easy to verify that

$$\boldsymbol{V}\boldsymbol{b}_{13}' = \boldsymbol{\Sigma}_{31} - \frac{\boldsymbol{b}_{32}}{b_{12}}\Sigma_{11} - \boldsymbol{b}_{31}b_{11} + \frac{\boldsymbol{b}_{32}}{b_{12}}b_{11}^2 \tag{79}$$

Lemma 1. *Matrix $\boldsymbol{V}$ is full rank.*

Proof. It is easy to construct a transformation matrix $\boldsymbol{A} = \begin{bmatrix} I & 0 & \mathbf{0} \\ 0 & I & \mathbf{0} \\ -\frac{\boldsymbol{b}_{32}}{b_{12}} & \mathbf{0} & \boldsymbol{I} \end{bmatrix}$.

Note that $\text{var}\left(\boldsymbol{A}\boldsymbol{u}_t\right) = \boldsymbol{A}\boldsymbol{\Sigma}\boldsymbol{A}'$ is full rank as $\boldsymbol{A}$ and $\boldsymbol{\Sigma}$are full rank. Further notice that

$$\boldsymbol{A}\boldsymbol{u}_t = \boldsymbol{A}\boldsymbol{B}\varepsilon_t = \begin{bmatrix} b_{11} & b_{12} & \boldsymbol{b}_{13} \\ b_{21} & b_{22} & \boldsymbol{b}_{23} \\ -\frac{\boldsymbol{b}_{32}}{b_{12}}b_{11} + \boldsymbol{b}_{31} & \mathbf{0} & \boldsymbol{V} \end{bmatrix} \varepsilon_t$$

Since $\text{var}(\boldsymbol{\varepsilon}_t) = \boldsymbol{I}$ is full rank, $\boldsymbol{AB}$ must be full rank. Therefore $\boldsymbol{V}$ must be full rank. ☐

From above equation 79 we can know that

$$\boldsymbol{b}_{13}\boldsymbol{V}'\boldsymbol{V}\boldsymbol{b}_{13}' = \left(\boldsymbol{\Sigma}_{31} - \frac{\boldsymbol{b}_{32}}{b_{12}}\Sigma_{11} - \boldsymbol{b}_{31}b_{11} + \frac{\boldsymbol{b}_{32}}{b_{12}}b_{11}^2\right)' \left(\boldsymbol{\Sigma}_{31} - \frac{\boldsymbol{b}_{32}}{b_{12}}\Sigma_{11} - \boldsymbol{b}_{31}b_{11} + \frac{\boldsymbol{b}_{32}}{b_{12}}b_{11}^2\right)$$

Then we define $\boldsymbol{Q} = \boldsymbol{V}\boldsymbol{V}'$.

Lemma 2. *$\boldsymbol{Q}$ is full rank and $\boldsymbol{Q}^{-1}$ exists.*

Proof. From Lemma 1 we know that matrix $\boldsymbol{V}$ is a square matrix and is full rank, too. Therefore $\boldsymbol{Q}$ is a square matrix and full rank since demote$\boldsymbol{Q} = \boldsymbol{V}\boldsymbol{V}'$. Then we can yield the conclusion that $\boldsymbol{Q}^{-1}$ exists. ☐

Lemma 3. *Given the $\boldsymbol{Q}$ we would have the relationship that $\boldsymbol{V}'\boldsymbol{Q}^{-1}\boldsymbol{V} = \boldsymbol{I}$.*

Proof. This is easy to prove. It is worth to notice that $\boldsymbol{V}'\boldsymbol{Q}^{-1}\boldsymbol{V}$ is just a projection matrix on the column space of $\boldsymbol{V}'$. Since $\boldsymbol{Q}$ and $\boldsymbol{V}$ are full rank, we must have a complete mapping which means $\boldsymbol{V}'\boldsymbol{Q}^{-1}\boldsymbol{V} = \boldsymbol{I}$. ☐

Because of $\boldsymbol{V}'\boldsymbol{Q}^{-1}\boldsymbol{V} = \boldsymbol{I}$, we would have

$$\boldsymbol{b}_{13}\boldsymbol{b}_{13}' = \boldsymbol{b}_{13}\boldsymbol{V}'\boldsymbol{Q}^{-1}\boldsymbol{V}\boldsymbol{b}_{13}' = \left(\boldsymbol{\Sigma}_{31} - \frac{\boldsymbol{b}_{32}}{b_{12}}\Sigma_{11} - \boldsymbol{b}_{31}b_{11} + \frac{\boldsymbol{b}_{32}}{b_{12}}b_{11}^2\right)' \boldsymbol{Q}^{-1} \left(\boldsymbol{\Sigma}_{31} - \frac{\boldsymbol{b}_{32}}{b_{12}}\Sigma_{11} - \boldsymbol{b}_{31}b_{11} + \frac{\boldsymbol{b}_{32}}{b_{12}}b_{11}^2\right)$$

E.2 Estimation of coefficient vectors under lower triangle assumption.

Write the partition of covariance $\boldsymbol{\Sigma}_{\boldsymbol{mu}'}$

$$\boldsymbol{\Sigma}_{\boldsymbol{mu}'} = \begin{bmatrix} \Sigma_{mu'_{11}} & \Sigma_{mu'_{12}} & \boldsymbol{\Sigma}_{\boldsymbol{mu}'_{13}} \\ \Sigma_{mu'_{21}} & \Sigma_{mu'_{22}} & \boldsymbol{\Sigma}_{\boldsymbol{mu}'_{23}} \end{bmatrix}$$

Based on $\boldsymbol{\Phi}\boldsymbol{\beta}_1' = \boldsymbol{\Sigma}_{\boldsymbol{mu}'}$ and $\boldsymbol{\Phi} = \begin{bmatrix} \alpha^s & 0 \\ \alpha^p & \alpha^d \end{bmatrix}$ I can write

$$\alpha^p \begin{bmatrix} b_{11} & b_{21} & \boldsymbol{b}_{31}' \end{bmatrix} + \alpha^d \begin{bmatrix} b_{12} & b_{22} & \boldsymbol{b}_{32}' \end{bmatrix} = \begin{bmatrix} \Sigma_{mu'_{21}} & \Sigma_{mu'_{22}} & \boldsymbol{\Sigma}_{\boldsymbol{mu}'_{23}} \end{bmatrix} \tag{80}$$

Using the first linear relationship I can rule out α^p as

$$\alpha^p = \Sigma_{mu'_{21}}\frac{1}{b_{11}} - \alpha^d\frac{b_{12}}{b_{11}} \tag{81}$$

Plugging above equation into equation 80 yields

$$\Sigma_{mu'_{21}}\frac{1}{b_{11}}\begin{bmatrix} b_{11} & b_{21} & \boldsymbol{b}_{31}' \end{bmatrix} + \alpha^d \begin{bmatrix} b_{12} - b_{12} & b_{22} - b_{12}\frac{b_{21}}{b_{11}} & \boldsymbol{b}_{32}' - b_{12}\frac{\boldsymbol{b}_{31}'}{b_{11}} \end{bmatrix} = \begin{bmatrix} \Sigma_{mu'_{21}} & \Sigma_{mu'_{22}} & \boldsymbol{\Sigma}_{\boldsymbol{mu}'_{23}} \end{bmatrix}$$

Then use the linear restriction at the second column to eliminate α^d as

$$\alpha^d = \frac{\Sigma_{mu'_{22}} - \Sigma_{mu'_{21}} \frac{b_{21}}{b_{11}}}{b_{22} - b_{12}\frac{b_{21}}{b_{11}}}$$

which will yield the final valuable restriction

$$\left(\Sigma_{mu'_{22}} - \Sigma_{mu'_{21}} \frac{b_{21}}{b_{11}}\right) \left[\frac{\boldsymbol{b}'_{32} - b_{12}\frac{\boldsymbol{b}'_{31}}{b_{11}}}{b_{22} - b_{12}\frac{b_{21}}{b_{11}}}\right] = \left[\boldsymbol{\Sigma}_{\boldsymbol{mu}'_{23}} - \Sigma_{mu'_{21}} \frac{\boldsymbol{b}'_{31}}{b_{11}}\right]$$

Since $\frac{\left[\boldsymbol{\Sigma}_{\boldsymbol{mu}'_{23}} - \Sigma_{mu'_{21}} \frac{\boldsymbol{b}'_{31}}{b_{11}}\right]}{\Sigma_{mu'_{22}} - \Sigma_{mu'_{21}} \frac{b_{21}}{b_{11}}}$ is estimable, I can estimate $\left[\begin{array}{ccc} b_{12} & b_{22} & \boldsymbol{b}'_{32} \end{array}\right]$ as long as given b_{22} and b_{12}.

To estimate $\left[\frac{\boldsymbol{b}'_{32} - b_{12}\frac{\boldsymbol{b}'_{31}}{b_{11}}}{b_{22} - b_{12}\frac{b_{21}}{b_{11}}}\right]$ I can run the IV regression such that

$$\boldsymbol{u}_{3t} - \frac{\boldsymbol{b}_{31}}{b_{11}} u_{1t} = \gamma + \theta \left(u_{2t} - \widehat{\frac{b_{21}}{b_{11}}} u_{1t}\right) + \zeta_t$$

where $\left(u_{2t} - \widehat{\frac{b_{21}}{b_{11}}} u_{1t}\right)$ comes from the instrument estimation

$$u_{2t} - \frac{b_{21}}{b_{11}} u_{1t} = \tau + \psi m_{2t} + \eta_t$$

To estimate b_{22} and b_{12} I use the covariance matrix $\boldsymbol{\Sigma}$ under a convention to sign. Write the $\boldsymbol{B}$ into partition

$$\boldsymbol{B} = \left[\begin{array}{cc} \boldsymbol{s}_{11} & \boldsymbol{s}_{12} \\ \boldsymbol{s}_{21} & \boldsymbol{s}_{22} \end{array}\right]$$

$$\boldsymbol{\Sigma} = \left[\begin{array}{cc} \boldsymbol{\sigma}_{11} & \boldsymbol{\sigma}_{12} \\ \boldsymbol{\sigma}_{21} & \boldsymbol{\sigma}_{22} \end{array}\right]$$

where $\boldsymbol{s}_{11}$ is the 2-by-2 matrix. Then I can use the relationship

$$\boldsymbol{s}_{11}\boldsymbol{s}'_{11} = \boldsymbol{\sigma}_{11} - \boldsymbol{s}_{12}\boldsymbol{s}'_{12} \tag{82}$$

$$\boldsymbol{s}_{12}\boldsymbol{s}'_{12} = \left(\boldsymbol{\sigma}_{21} - \boldsymbol{s}_{21}\boldsymbol{s}_{11}^{-1}\boldsymbol{\sigma}_{11}\right)' \boldsymbol{Q}^{-1} \left(\boldsymbol{\sigma}_{21} - \boldsymbol{s}_{21}\boldsymbol{s}_{11}^{-1}\boldsymbol{\sigma}_{11}\right)$$

$$\boldsymbol{Q} = \boldsymbol{s}_{21}\boldsymbol{s}_{11}^{-1}\boldsymbol{\sigma}_{11}\left(\boldsymbol{s}_{21}\boldsymbol{s}_{11}^{-1}\right)' - \left(\boldsymbol{\sigma}_{21}\left(\boldsymbol{s}_{21}\boldsymbol{s}_{11}^{-1}\right)' + \boldsymbol{s}_{21}\boldsymbol{s}_{11}^{-1}\boldsymbol{\sigma}'_{21}\right) + \boldsymbol{\sigma}_{22}$$

where $\boldsymbol{s}_{21}\boldsymbol{s}_{11}^{-1}$ is estimated by iv regression.

E.3 Bayesian estimation of coefficient in IV

Because of equation 24 I can write it as

$$\boldsymbol{\Phi}\boldsymbol{\beta}_1'\boldsymbol{\beta}_1\boldsymbol{\Phi}' = \boldsymbol{\Sigma}_{\boldsymbol{mu}'}\boldsymbol{\Sigma}_{\boldsymbol{mu}'}'$$

Taking Cholesky decomposition of Σ yields

$$\boldsymbol{B}\boldsymbol{B}' = \Sigma = \boldsymbol{\Sigma}_{tr}\Gamma\Gamma'\boldsymbol{\Sigma}_{tr}' = \boldsymbol{\Sigma}_{tr}\boldsymbol{\Sigma}_{tr}'$$

where $\boldsymbol{\Sigma}_{tr}$ is a lower triangle matrix and Γ is an orthogonal matrix such that $\Gamma \in \mathcal{O}(n)$ and $\boldsymbol{B} = \boldsymbol{\Sigma}_{tr}\Gamma$

Since $\boldsymbol{\beta}_1'\boldsymbol{\beta}_1 = e_1'\boldsymbol{B}'\boldsymbol{B}e_1 = e_1'\Gamma'\boldsymbol{\Sigma}_{tr}'\boldsymbol{\Sigma}_{tr}\Gamma e_1$ and I denote $D = \boldsymbol{\Sigma}_{tr}'\boldsymbol{\Sigma}_{tr}$

Then above equation can be write as

$$(\boldsymbol{\Phi}e_1'\Gamma' \otimes \boldsymbol{\Phi}e_1'\Gamma')\,\mathrm{vec}\,(D) = \mathrm{vec}\,(\boldsymbol{\Sigma}_{\boldsymbol{mu}'}\boldsymbol{\Sigma}_{\boldsymbol{mu}'}')$$

Therefore this can yield

$$(\boldsymbol{\Phi}e_1'\Gamma' \otimes \boldsymbol{\Phi}e_1'\Gamma') = \mathrm{vec}\,(\boldsymbol{\Sigma}_{\boldsymbol{mu}'}\boldsymbol{\Sigma}_{\boldsymbol{mu}'}')\,[\mathrm{vec}\,(D)]'\,\Xi^{-1}$$

where

$$\Xi = \mathrm{vec}\,(D)\,[\mathrm{vec}\,(D)]'$$

Then by right multiplying $\mathrm{vec}\,(I)$ on both side I can rearrange this equation to

$$(\boldsymbol{\Phi}e_1'\Gamma' \otimes \boldsymbol{\Phi}e_1'\Gamma')\,\mathrm{vec}\,(I) = \mathrm{vec}\,(\boldsymbol{\Sigma}_{\boldsymbol{mu}'}\boldsymbol{\Sigma}_{\boldsymbol{mu}'}')\,[\mathrm{vec}\,(D)]'\,\Xi^{-1}\mathrm{vec}\,(I)$$

$$\mathrm{vec}\,(\boldsymbol{\Phi}e_1'\Gamma'\Gamma e_1\boldsymbol{\Phi}') = \mathrm{vec}\,(\boldsymbol{\Phi}\boldsymbol{\Phi}') = \mathrm{vec}\,(\boldsymbol{\Sigma}_{\boldsymbol{mu}'}\boldsymbol{\Sigma}_{\boldsymbol{mu}'}')\,[\mathrm{vec}\,(D)]'\,\Xi^{-1}\mathrm{vec}\,(I)$$

As long as Ξ is invertable.

Otherwise write equation 24 as

$$\boldsymbol{\Phi}\boldsymbol{s}_{11}' = \boldsymbol{\Sigma}_{\boldsymbol{mu}_1'}$$

Because $\boldsymbol{s}_{11}'$, $\boldsymbol{\Sigma}_{\boldsymbol{mu}'1}$ and $\boldsymbol{\Phi}$ are full rank,

$$\boldsymbol{\Phi} = \boldsymbol{\Sigma}_{\boldsymbol{mu}_1'}\boldsymbol{s}_{11}'^{-1}$$

$$\boldsymbol{\Phi}\boldsymbol{\Phi}' = \boldsymbol{\Sigma}_{\boldsymbol{mu}_1'}\boldsymbol{s}_{11}'^{-1}\boldsymbol{s}_{11}^{-1}\boldsymbol{\Sigma}_{\boldsymbol{mu}_1'}'$$

$$\boldsymbol{\Phi}\boldsymbol{\Phi}'\boldsymbol{\Sigma}_{\boldsymbol{mu}_1'}'^{-1} = \boldsymbol{\Sigma}_{\boldsymbol{mu}_1'}\boldsymbol{s}_{11}'^{-1}\boldsymbol{s}_{11}^{-1}$$

$$\boldsymbol{\Phi}\boldsymbol{\Phi}'\boldsymbol{\Sigma}_{\boldsymbol{mu}_1'}'^{-1}\boldsymbol{s}_{11}\boldsymbol{s}_{11}' = \boldsymbol{\Sigma}_{\boldsymbol{mu}_1'}$$

$$\boldsymbol{\Phi}\boldsymbol{\Phi}' = \boldsymbol{\Sigma}_{\boldsymbol{m}\boldsymbol{u}_1'} \left(\boldsymbol{\Sigma}_{\boldsymbol{m}\boldsymbol{u}_1'}^{\prime -1} \boldsymbol{s}_{11} \boldsymbol{s}_{11}' \right)^{-1}$$

where $\boldsymbol{s}_{11}^{-1} \boldsymbol{s}_{11}^{\prime -1}$ can be estimated from equation 82

Firstly I show how to derive the conditional likelihood $p \left(M \middle| Y, X, Q, B, u, \sigma_m \right)$

Following Giacomini et al. (2021) I can write the stochastic variable jointly such that

$$\begin{bmatrix} Y_t - \Psi X_t \\ m_t - \nu \end{bmatrix} = \begin{bmatrix} B & 0 \\ \boldsymbol{\Phi}^0 & \sigma_m \end{bmatrix} \begin{bmatrix} \boldsymbol{\varepsilon}_t \\ \zeta_t \end{bmatrix} \tag{83}$$

where

$$\boldsymbol{\Phi}^0 = \begin{bmatrix} \boldsymbol{\Phi} & \mathbf{0} \end{bmatrix}$$

that is based on the assumption of instrument variable, the last $n - k$ element in $\boldsymbol{\varepsilon}_t$ do not have any cross effect with m_t[53]

Therefore we would have

$$M \middle| Y, X, Q, B, u, \sigma_m \sim N \left(\mu_{m|u}, \Sigma_{m|u} \right) \tag{84}$$

where

$$\begin{aligned} \mu_{m|u} &= E \left(\boldsymbol{\Phi}^0 \boldsymbol{\varepsilon}_t + \sigma_m \zeta_t + \nu \right) + \boldsymbol{\Phi}^0 B' \left(\boldsymbol{\Sigma} \right)^{-1} \left(u_t - E \left(B \boldsymbol{\varepsilon}_t \right) \right) \\ &= \boldsymbol{\Sigma}_{\boldsymbol{m}\boldsymbol{u}'} \boldsymbol{\Sigma}^{-1} u_t + \nu \end{aligned}$$

and

$$\begin{aligned} \Sigma_{m|u} &= \boldsymbol{\Phi}^0 \boldsymbol{\Phi}^{0\prime} + \sigma_m \sigma_m' - \boldsymbol{\Phi}^0 B' \left(B B' \right)^{-1} B \boldsymbol{\Phi}^0 \\ &= \sigma_m \sigma_m' \end{aligned}$$

since $B' \left(BB' \right)^{-1} B$ is a projection matrix on the column space of B and it is a full rank matrix, this term will be an identity matrix.

To calculate the prior distribution of σ_m I use the regression

$$m_t = \nu + P u_t + \zeta_t$$

where it is straightforward to prove that

$$P = \boldsymbol{\Phi}^0 B$$

ζ_t and ν are the same variables comparing to equation 83.

[53] In fact this step is necessary otherwise the $\boldsymbol{\beta}_2$ effect will eliminate by $\mathbf{0}$ in $\boldsymbol{\Phi}^0$ and I need more degree of freedom to identify. For instance, $\Sigma_{m_t|u_t} = BB' - B\boldsymbol{\Phi}^{0\prime} \left(\boldsymbol{\Phi}^0 \boldsymbol{\Phi}^{0\prime} + \sigma_m \sigma_m' \right)^{-1} \boldsymbol{\Phi}^0 B'$, $\mathbf{0}$ in $\boldsymbol{\Phi}^0$ will cause $B\boldsymbol{\Phi}^{0\prime} = \boldsymbol{\beta}_1 \boldsymbol{\Phi}$ such that $\boldsymbol{\beta}_2$ has no effect.

Then because the posterior distribution of the parameter can be write as

$$\begin{aligned} p\left(B, \Sigma, Q, \sigma_m \sigma_m' \middle| Y, M\right) &\propto p\left(Y, M, B, \Sigma, Q, \sigma_m \sigma_m'\right) p\left(B, \Sigma, Q, \sigma_m \sigma_m'\right) \\ &\propto p\left(M \middle| Y, Q^i, B, \Sigma, \sigma_m^i\right) p\left(B, \Sigma \middle| Y\right) \end{aligned}$$

The first term $p\left(M \middle| Y, Q^i, B, \Sigma, \sigma_m^i\right)$ can be calculated through the distribution 84. The second term $p\left(B, \Sigma \middle| Y\right)$ can also easily be calculated through the normal-inverse-Wishart family of distributions which is proposed by Arias et al. (2018).

Then draw rotation matrix Q based on the algorithm discussed below. After drawing a set of Q, Ω^Q, I can solve the posterior distribution $p\left(Q \middle| Y, X, M, B, u, \Sigma\right)$where ε_t and ζ_t follow two independent standard normal distribution.

E.3.1 Algorithm used to produce figure 5

E.3.2 Alternative algorithm

Algorithm 1 Draw Q from $Q\big|Y, X, M, B, u, \sigma_m$

1. Draw B^i and Σ^i from the NIW $(\nu, \Phi, \Psi, \Omega)$.
2. Accept B^i and Σ^i based on the probability

$$\rho = \min\left\{\frac{\mathcal{L}\left(M\big|Y, Q^{i-1}, B^i, \Sigma^i, \sigma_m^{i-1}\right)}{\mathcal{L}\left(M\big|Y, Q^{i-1}, B^{i-1}, \Sigma^{i-1}, \sigma_m^{i-1}\right)}, 1\right\}$$

3. Based on B^i and Σ^i to draw new residual and $\boldsymbol{\Sigma_{mu'}}$
4. Draw Q^i based on the Theorem 9 in Rubio-Ramirez et al. (2010).
5. Draw $\sigma_m^i \sigma_m'^i$ from $IW_k(S, T+\tau)$ where τ is the prior degree of freedom. Denote V the prior variance of $\boldsymbol{\Phi}$,

$$S = (M - PU)(M - PU)' + S_0 + \hat{A}UU'\hat{A}' + A^* V^{-1} A^{*\prime} - \bar{A}\left(V^{-1} + UU'\right)\bar{A}'$$

where A^* is the prior mean of $\boldsymbol{\Phi}$;V is the prior covariance matrix of $\boldsymbol{\Phi}$

$$\hat{A} = MU'(UU')^{-1}$$

$$A^* = \begin{bmatrix} \nu & \boldsymbol{\Phi}_{tr} Q^{i-1} \end{bmatrix}$$

$$V = \alpha^* I_{k+1}$$

$$\bar{A} = \left(A^* V^{-1} + MU'\right)\left(V^{-1} + UU'\right)^{-1}$$

6. calculate

$$\rho = \min\left\{\frac{\mathcal{L}\left(M\big|Y, Q^i, B^i, \Sigma^i, \sigma_m^i\right)}{\mathcal{L}\left(M\big|Y, Q^{i-1}, B^i, \Sigma^i, \sigma_m^{i-1}\right)}, 1\right\}$$

and I take $\boldsymbol{\rho}$ as the probability that retains new Q^i, otherwise $Q^i = Q^{i-1}$.

$$\mathcal{L}\left(M\big|Y, Q^i, B, \Sigma, \sigma_m^i\right) \propto \left\{-\frac{kT}{2}\log(2\pi) - \frac{1}{2}\log\left(\left|I_T \otimes \sigma_m \sigma_m'\right|\right) - \frac{1}{2}\left[\text{vec}(M) - (U' \otimes I_{k+1})\,\text{vec}(Z)\right]'\left[I_T \otimes \left(\sigma_m \sigma_m'\right)^{-1}\right]\left[\text{vec}(M) - (U' \otimes I_{k+1})\,\text{vec}(Z)\right]\right\}$$

where

$$M \equiv \begin{bmatrix} m_1 & m_2 & \ldots & m_{T-1} & m_T \end{bmatrix}$$

$$U \equiv \begin{bmatrix} u_1^0 & u_2^0 & \ldots & u_{T-1}^0 & u_T^0 \end{bmatrix}$$

$$u_t^0 = [1, u_t']'$$

$$Z = \begin{bmatrix} \nu & \boldsymbol{\Phi}_{tr} Q^i \left(\boldsymbol{\Phi}_{tr} Q^{i-1}\right)^{-1} \boldsymbol{\Sigma_{mu'}} \boldsymbol{\Sigma}^{-1} \end{bmatrix}$$

ν can be calculated by run regression of m_t on u_t^0 as

$$M = PU + \zeta$$

Algorithm 2 Draw Q from $Q \middle| Y, X, M, B, u, \sigma_m$

1. Estimate B and Σ from the standard reduced-form regression.
2. Draw Q^i based on the Theorem 9 in Rubio-Ramirez et al. (2010).
3. Draw $\sigma_m^i \sigma_m'^i$ from $IW_k(S, T+\tau)$ where τ is the prior degree of freedom. Denote V the prior variance of $\mathbf{\Phi}$,

$$S = (M - PU)(M - PU)' + S_0 + \hat{A}UU'\hat{A}' + A^* V^{-1} A^{*\prime} - \bar{A}\left(V^{-1} + UU'\right)\bar{A}'$$

where A^* is the prior mean of $\mathbf{\Phi}$;V is the prior covariance matrix of $\mathbf{\Phi}$

$$\hat{A} = MU'(UU')^{-1}$$

$$A^* = \begin{bmatrix} \nu & \mathbf{\Phi}_{tr} Q^{i-1} \end{bmatrix}$$

$$V = \alpha^* I_{k+1}$$

$$\bar{A} = \left(A^* V^{-1} + MU'\right)\left(V^{-1} + UU'\right)^{-1}$$

4. calculate

$$\rho = \min\left\{ \frac{\mathcal{L}\left(M \middle| Y, Q^i, B, \Sigma, \sigma_m^i\right)}{\mathcal{L}\left(M \middle| Y, Q^{i-1}, B, \Sigma, \sigma_m^{i-1}\right)}, 1 \right\}$$

and I take $\boldsymbol{\rho}$ as the probability that retains new Q^i, otherwise $Q^i = Q^{i-1}$.

$$\mathcal{L}\left(M \middle| Y, Q^i, B, \Sigma, \sigma_m^i\right) \propto \Big\{ -\frac{kT}{2}\log(2\pi) - \frac{1}{2}\log\left(\left|I_T \otimes \sigma_m \sigma_m'\right|\right) - \frac{1}{2}\left[\text{vec}(M) - (U' \otimes I_{k+1})\,\text{vec}(Z)\right]' \left[I_T \otimes (\sigma_m \sigma_m')^{-1}\right] \left[\text{vec}(M) - (U' \otimes I_{k+1})\,\text{vec}(Z)\right]\Big\}$$

where

$$M \equiv \begin{bmatrix} m_1 & m_2 & \ldots & m_{T-1} & m_T \end{bmatrix}$$

$$U \equiv \begin{bmatrix} u_1^0 & u_2^0 & \ldots & u_{T-1}^0 & u_T^0 \end{bmatrix}$$

$$u_t^0 = [1, u_t']'$$

$$Z = \begin{bmatrix} \nu & \mathbf{\Phi}_{tr} Q^i \left(\mathbf{\Phi}_{tr} Q^{i-1}\right)^{-1} \mathbf{\Sigma}_{\boldsymbol{mu'}} \mathbf{\Sigma}^{-1} \end{bmatrix}$$

ν can be calculated by run regression of m_t on u_t^0 as

$$M = PU + \zeta$$

Algorithm 3 Draw Q from $Q|Y, X, M, B, u, \sigma_m$

1. Draw $B^{i-1,1}$ and $\Sigma^{i-1,1}$ from the NIW $(\nu, \Phi, \Psi, \Omega)$.

2. Accept $B^{i-1,1}$ and $\Sigma^{i-1,1}$ based on the probability

$$\rho = \min\left\{\frac{\mathcal{L}\left(M\middle|Y, Q^{i-1}, B^{i-1,1}, \Sigma^{i-1,1}, \sigma_m^{i-1}\right)}{\mathcal{L}\left(M\middle|Y, Q^{i-1}, B^{i-1,0}, \Sigma^{i-1,0}, \sigma_m^{i-1}\right)}, 1\right\}$$

3. iterate back to step 1 and draw new $B^{i-1,j}$ and $\Sigma^{i-1,j}$. After burn-in step, continue to draw N step and take B^i and Σ^i as their mean.

4. Based on B^i and Σ^i to draw new residual and $\boldsymbol{\Sigma_{mu'}}$

5. Draw Q^i based on the Theorem 9 in Rubio-Ramirez et al. (2010).

6. Draw $\sigma_m^i \sigma_m'^i$ from $IW_k(S, T+\tau)$ where τ is the prior degree of freedom. Denote V the prior variance of $\boldsymbol{\Phi}$,

$$S = (M - PU)(M - PU)' + S_0 + \hat{A}UU'\hat{A}' + A^* V^{-1} A^{*\prime} - \bar{A}\left(V^{-1} + UU'\right)\bar{A}'$$

   where A^* is the prior mean of $\boldsymbol{\Phi}$;V is the prior covariance matrix of $\boldsymbol{\Phi}$

$$\hat{A} = MU'(UU')^{-1}$$

$$A^* = \left[\begin{array}{cc} \nu & \boldsymbol{\Phi}_{tr} Q^{i-1} \end{array}\right]$$

$$V = \alpha^* I_{k+1}$$

$$\bar{A} = \left(A^* V^{-1} + MU'\right)\left(V^{-1} + UU'\right)^{-1}$$

7. calculate

$$\rho = \min\left\{\frac{\mathcal{L}\left(M\middle|Y, Q^{i}, B^{i}, \Sigma^{i}, \sigma_m^{i}\right)}{\mathcal{L}\left(M\middle|Y, Q^{i-1}, B^{i}, \Sigma^{i}, \sigma_m^{i-1}\right)}, 1\right\}$$

   and I take $\boldsymbol{\rho}$ as the probability that retains new Q^i, otherwise $Q^i = Q^{i-1}$.

$$\mathcal{L}\left(M\middle|Y, Q^i, B, \Sigma, \sigma_m^i\right) \propto \left\{-\frac{kT}{2}\log(2\pi) - \frac{1}{2}\log\left(\left|I_T \otimes \sigma_m \sigma_m'\right|\right)\right.$$
$$-\frac{1}{2}\left[\operatorname{vec}(M) - (U' \otimes I_{k+1})\operatorname{vec}(Z)\right]'\left[I_T \otimes (\sigma_m \sigma_m')^{-1}\right]$$
$$\left.\left[\operatorname{vec}(M) - (U' \otimes I_{k+1})\operatorname{vec}(Z)\right]\right\}$$

   where

$$M \equiv \left[\begin{array}{ccccc} m_1 & m_2 & \dots & m_{T-1} & m_T \end{array}\right]$$

$$U \equiv \left[\begin{array}{ccccc} u_1^0 & u_2^0 & \dots & u_{T-1}^0 & u_T^0 \end{array}\right]$$

$$u_t^0 = [1, u_t']'$$

$$Z = \left[\begin{array}{cc} \nu & \boldsymbol{\Phi}_{tr} Q^i \left(\boldsymbol{\Phi}_{tr} Q^{i-1}\right)^{-1} \boldsymbol{\Sigma_{mu'}} \boldsymbol{\Sigma}^{-1} \end{array}\right]$$

   ν can be calculated by run regression of m_t on u_t^0 as

$$M = PU + \zeta$$

8. Back to step 1