Quantitative Easing and the Redistribution Channel

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Abstract

In this paper, I document two opposite channels that the Quantitative Easing (QE) can influence the wealth inequality in Unite States. On the one hand, an economic recovery, facilitated by QE, serves to reduce inequality through general equilibrium effects and labor market dynamics by elevating the lower-income cohorts to middle-class status (MPC channel). On the other hand, diminished returns on financial assets have a regressive effect, demoting middle-class households to lower income levels (MPS channel). The net impact of QE on wealth inequality, therefore, is contingent upon the relative strengths of these opposing forces. Then, by setting up a general equilibrium model with heterogeneous household and financial frictions, I demonstrate that the MPC channel overwhelms the MPS channel and the QE increases wealth inequality significantly. Meanwhile, I further show that the macroprudential policy can attenuate the exacerbation of inequality notably.

JEL classification: C11, E21, E30, E52, E58

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*PhD candidate in Hong Kong University of Science and Technology. Email:xshiah@connect.ust.hk. Firstly, I want to thank my supervisor Prof Jenny XU for her full support, from choice of topic and literature review to intuition, model setting and result analysis. I will not be able to finish my thesis without her help. Secondly I want to thank Prof Byoungchan LEE who also gave me great assistance. I thank him for his discussion with me about my thesis, especially on literature, intuition, detailed model part and quantitative solution methods. Thirdly I want to thank my parents for their mental and financial support. Fourthly I want to thank senior colleagues Yue ZHOU and Jun YU who encouraged and urged me to work on my thesis. I thank them for accompanying with me every day when I struggled on my thesis and encouraging me throughtout the process.

1 Introduction

There are many instruments in the arsenal of central banks such as the open-market operation, interest rate setting, reserve requirement, etc. Normally, we call them conventional monetary policies, since they have been used for several decades. However, starting at the recession between 2007 and 2009, conventional monetary policies ceased to be used when interest rate decreased to nearly zero. The central banks could not change it anymore. Because of the ZLB different central banks proposed unconventional monetary policies like negative interest rate, forward guidance (McKay et al. (2016),Angeletos and Lian (2018), Hagedorn et al. (2019)), and quantitative easing (Gagnon et al. (2011), Wu and Xia (2016), Swanson (2017)). This change of monetary policy regime led to a heated discussion about the real effect of unconventional monetary policy. In addition to the real effect, the inequality problem is another important aspect. If a policy stimulates the economy and aggravates the inequality problem at the same time, it cannot be called a successful policy.Has the QE caused the inequality problem? What about the quantitative easing(QE) policy? Will its effect be small like Carrera et al. (2014) claim or large, as expressed in the balance sheet effect, like claims in De Ferra et al. (2020)? My paper will try to answer these questions.

There are many theoretical literatures focusing on the effects of different monetary policies, conventional or unconventional on the economy. Most of them shed light on the transmission mechanism to emphasize the real effect. However, this literature simply use a representative-agent model attempting to analyze the effect of monetary policy, such as Yogo (2004) and Canzoneri et al. (2007), which only examine the aggregate effect of the QE. This literature can describe part of the real world, but it cannot address the wealth redistribution and inequality problem, while sometimes allocating the cake evenly is more important than making the cake bigger. Though some papers pay attention to the distribution effects of monetary policy. such as Doepke and Schneider (2006), Sterk and Tenreyro (2018), Gornemann et al. (2016), and McKay et al. (2016), most of them use two agent model with manufacturer and counterfactual assumptions, for instance, hand-to-mouth households, to fit the model to real life. By contrast, the study in this paper use the newly developed algorithm by Achdou et al. (2017) and Kaplan et al. (2018), which is powerful in arguing and analyzing the true effect of the monetary policy on the economy. As discussed later, the canonical model cannot depict economic activates in household sector and may lead to incorrect conclusion and inconsistence with data. This paper use the continuous heterogeneous agent model, because predictions from the continuous model are consistent with data either in household sector or in financial institution sector.

Empirical literatures have argued that QE increased wealth inequality, such as Saiki and Frost (2014), Montecino et al. (2015), Yoshino et al. (2018), and Furceri et al. (2018). Some other literatures like Gambacorta et al. (2014) and Colciago et al. (2019), believe that economic recovery brought by QE will also decrease income inequality and that the overall effect will be mixed.

However, to my knowledge, few papers have studied that effect of QE on wealth inequality. We use a HANK model to specify the MPC channel through which the QE change the wealth inequality of household. Correspondingly, we use quantitative results to verify our intuitions as well as perform counterfactual experiments that help us understand QE policy more clearly. This paper also analyzes the interaction between QE policy and the macroprudential policy. It mainly focuses on redistribution and MPC channel responses to quantitative easing monetary policies which have not been clearly identified and quantitatively analyzed before in the literature. In addition to the distribution effect in the household sector, our study also embed financial sector into the model, not only to describe the QE on the wealth inequality more clearly, but also to explaine and match effects of QE on the financial market. Most of the heterogeneous household literature focus on demand-side effects, such as demand externalities under a demand-driven economy. Nevertheless, the supply side is also important to understand the effect of a policy. Another contribution of our paper is that it provides a circumstance where the setting on the supply side is pivotal for analyzing policy effects. Therefore, this paper can answer questions regarding wealth inequality raised earlier.

Suppose that the economy fell into a great recession and the short term interest rate has reached zero, where conventional monetary policies can no longer be used to stimulate the economy. The central bank proposes an unconventional monetary policy, QE, to stimulate the economy. It buys a large amount of the long-term assets and increases the demand for long-term bonds with higher price (lower interest rate). This drop in long-term interest rates will depress the demand of investors who invest in long-term bonds. In other words, the central bank uses a large amount of purchasing to crowd out the long-term investors who put money in short-term bond instead now, which increases market liquidity. The insurance companies, pension funds, arbitrageurs, and financial banks who provide packaged financial products like trust, will be influenced by this policy and get more money back. Because of the drop in expected income, they will adjust the interest rate that they pay to the wealthy and middle-income depositors. This adjustment will decrease household demand to hold financial assets and decrease the demand for bonds issued by financial institutions. From another perspective, the decrease in demand for bonds will shrink financial institutions' asset volume (since their asset are the bonds they hold). Since their objective is to maximize their equity, they will in turn try to decrease the liability side by decreasing the investment return of households to push them to drawback some assets. This drawn-back money is part of the source of market liquidity. Furthermore, through the general equilibrium effect, this increased liquidity will be connected with an increase in consumption, which in turn increases production, labor demand, and wages, which lift the economy out of recession. The hand-to-mouth households on the demand side will not be influenced any more, as long as short-term interest rates have touched the zero boundary. They do not hold any assets and any adjustment to the asset part will not directly affect them. They are hand-to-mouth, and they do not have anything else to loss. However, the middle class and wealthy households will indeed be influenced, as they hold assets.

Paying greater attention to the effect on middle class and wealthy households more is important. One can use a two-agent model to match part of the inequality problems. But in this case, household will be divided into poor(this is hand-to-mouth households) and non-poor households. Yet the majority of the population consist of poor and middle-income households who have some assets and have sufficient income for consumption. It is easy to get a result from the TANK such that hand-to-mouth households are not directly affected by QE, since they have nothing to lose any more while the non-poor households will lose and it is their loss in asset holding that increases liquidity and stimulates the economy. In other words, the model suggests that the inequality problem is attenuated after QE. However, this is not the case. In fact, the influence of QE on asset holding within non-poor households is not proportional. The rich have a high consumption level, which causes their MPC to remain low. Even when their expected asset return decreases, they will not draw some assets back to consume given a MPC low enough. Middle income households, however, act differently, as they do not have a low-enough MPC, and they will withdraw some assets back to consume because of a decreased expected-return rate for assets. In other words, there is a twist of the asset level in non-poor households under the QE effect. When asset returns decrease, the rich will not put more money into their assets, not will they withdraw the assets either. Middle-income households are totally different, and they withdraw some part ,or even all of, the asset. This will cause a more severe inequality problem. QE does not rob the rich to poor to stimulate the economy, as the TANK predicts, but robes the middle income class to the poor and rich, to stimulate the economy. The channel proposed in this paper can be regard as the application of a first-order MPC channel, which has been widely used in analyzing different aspects of the economy.[1]

As financial technology becomes more and more popular, banks and other financial intermediates could alleviate the default and moral-hazard problem during the lending process and provide more liquidity to households, especially poor households to easily insure their income. This increase in the liquidity-injection power of financial market, resulting from the decrease in incompleteness, will attenuate the inequality result caused by QE via the precautionary saving channel discussed in Guerrieri and Lorenzoni (2017) and Bayer et al. (2019), or the MPC channel discussed in Korinek and Simsek (2016).

The increase in overall demand decreases the long-term yield, which is good news for firms, since they can borrow with lower costs by issuing long-term corporate bonds. They will face a lower cost of investment, to hire workers and increases production. Additionally, this new investment in capital and lower funding cost will increase the firms' share price[2].

[1] In addition to the first-order MPC channel that we focus on in this paper, there are also other mechanisms through which QE policy could influence the wealth inequality of household, but which are out of the scope of this paper. For instance, as empirically demonstrated in Guvenen et al. (2014) and theoretically discussed in Bilbiie (2018) and Acharya and Dogra (2020), the asymmetric precautionary saving channel causes asymmetric income risk between wealthy and poor people. There is also a second-order MPC channel, also called the "Intertemporal Keynesian Cross" that is discussed in Auclert et al. (2018) and Bilbiie (2019).

[2] If the firms use their securitized equity to borrow money in order to produce and invest in R&D, there will be a secondary supply-side expansionary effect of QE. However, this is beyond the scope of this paper.

The increased propensity for production will cause firms to hire more labor, by borrowing more from the bonds market and using this money to invest. Even though demand for long-term bonds has decreased, the supply side has risen and may dominate the demand side. The total amount of the bonds may increase eventually. Meanwhile, households will have more income to increase their consumption, which lifts the economy out of the recession due to the suply-side simulation in a general equilibrium setting. This supply-side effect is opposite to the demand-side effect, since it increases the volume of bonds as well as household wage income, especially in the case of poor workers. This counter effect, together with the demand-side effect, will decide the overall impact of QE on wealth inequality. If the demand-side effect is larger than supply-side, QE will increase wealth inequality and decrease the bond volume. Yet, if the supply-side effect is larger, QE will decrease bond volume and the change to wealth inequality will be unknown.

Furthermore, except the monetary policies, the policy makers have numbers of other instruments to adjust and control the economy activities such as fiscal policies and macroprudential policies. Normally they will not use a single type of policy to help the economy revive from recession. Thus the problems like cross effects between them naturally arise. This paper also contribute to the literature about the cross effects between monetary policies and macroprudential policies. We investigate the dilation effect of macroprudential policies on QE. We define the macroprudential policy here as a control on the leverage ratio which is same as Collard et al. (2017) used[3]. The tighten constraint on leverage ratio limits the financial institutions' ability to rise money to meet their demand in cash flow. The QE policy, which results in a decrease in long-term yield, will become more powerful because the financial institutions need to release more liquidity in the household sector and decrease the amount of financial products sold to households, which action can previously be replaced by raising money from banks or stock market.

All of the arguments and predictions above are supported by our quantitative results. We use an algorithm based on finite difference and upwind methods to solve the heterogeneous agent problem. We quantitatively get an increase in household assets' gini index from 0.4238 to 0.4245 which is consistent with either prediction or data. Except for the household sector, the quantitative results also show that the supply-side effect dominates the demand-side effect in financial sector. There finally comes out anisotropic changes in yield curve and bonds volume. The yield curve presents a U-shape change which is similar to the data. At the same time companies issue more long-term bonds and slightly less short-term bonds. Both direction and extent of this change are consistent with data as well.

The paper consists of five chapters. Chapter 2 mainly discusses the model and related proposition derived from it. Chapter 3 discusses the process used to process the numerical solutions and the related analysis. Then this research is concluded in Chapter 4.

[3]There are other types of the macorprudential policies such as interest rate subsidy in Korinek and Simsek (2016) and capital flow constraint in Schmitt-Grohe and Uribe (2012)

2 Baseline Model

In this section I introduce a simplified baseline general equilibrium model to take insight of the detailed way that QE affect the economy which will help a lot when we analyze the quantitative result later. The baseline model is simplified little but enough and can exhibit the detailed mechanism through which the QE influence the real economy. However, the contradictory effect of QE on inequality in demand and supply side will make the aggregate effect mixed and ambiguous. We cannot easily find out the total effect simply through investigate in the model but should base on the quantitative result.

2.1 Household

The household sector is a standard consumption setting as in Moll et al. (2019) that there is a mass of household holding different amount of asset which gives them related asset return each instant time along the continuous time. This asset is the financial products that they buy from the financial institution such as the insurance companies, pension funds, trust, funds based on bonds, ect. Since they hold the asset but not the simple risk free liquid bond, the asset return cannot go to zero as the setting of "long-run rate" in Kaplan and Violante (2014), "private interest rate" in Wu and Zhang (2019a) or "effective nominal rate" in Ray et al. (2019). Alternatively we can take them as the fixed asset that provideds a relatively fixed interest rate in long run such as the housing rent income, and pension fund as introduced in Kaplan and Violante (2014). The asset they buy from the financial institutions will ask them to pay the brokerage and transaction cost each time when they want to drawback or invest more in the institutions, as empirically observed by Edwards et al. (2004).

Meanwhile, each household can also deposit on the bank, which provides them safe storage with lower but not administration cost service. At the same time some households can also choose to borrow from the financial bank which will charge them a higher interest rate. Sometimes we also called it the interest rate spread. To avoid the Ponzi game, the bank, of course, will not allow the household to borrow as much as they, want and the lowest constraint is just their nature constraint while the central bank can definitely change the spread as well as the constraint as discussed in Guerrieri and Lorenzoni (2017).

For simplicity, we just assume that there is no employment and unemployment problem. It is easy to see when the economy goes down in recession time, the poorer you are, the more possibility you may lose your job and being unemployed. Therefore if we consider the unemployment problem we can anticipate easily, it will mitigate the dis-equality problem of QE via general equilibrium mechanism. When firm decides to invest more and produce more, it will choose to hire more workers to work and decrease the unemployment rate. This reduced unemployment rate will weaken the inequality problem as the poorer can work and earn more wage income now. Intuitively it is equivalent to the raised market wage, which our model

predicts[4]. Each household will get their income by provide their labor to the firm to produce. However their income gets from work is not the same as they will need to bear the idiosyncratic productivity shock which we can take as the bonus based wage income. If their productivity is higher, they will provide an effective higher labor supply to the firm who will pay them proportionally higher wage income based a base wage which decided by the market. Because the household have idiosyncratic income shock at the beginning of each instant period and cannot fully avoid the potential loss with incomplete risk sharing within the herd, even though the household can observe the shock at the beginning of each period, they still need precautionary saving to hedge the related income shock to smooth their consumption along the time line. They will choose to invest more in the asset since they fare to lose the high productivity sometimes later. They will deposit much in the financial bank since they are risk-averse agents and want to smooth their consumption in the lifetime. That will cause them to save more and make the conventional monetary policy harder to put an effect on the demand side, as discussed in Bayer et al. (2019). Meanwhile, households are still facing the intertemporal resource assignment problem since they can either consume 1 unit of good this period or save it in the bank account to earn a higher unit of good that could be consumed later. In addition to the standard intertemporal discount problem, we also assume that each household faces a probability of death at each period which makes the model closer to reality and mitigates some power of the precautionary saving. When they die, we will equally distribute their wealth to the lived household as liquidity asset (risk-free bonds). All new-born households will have the amount of risk-free bonds on average and zero financial assets.

For simplicity, we assume the a infinite lived household want to maximize the standard CRRA utility function based on the budget constraint. To solve the problem such that

$$Max\ E_0 \int_0^{\infty} e^{-(\rho+\zeta)t} u\left(c_t, \ell_t\right) dt$$

where ρ is the discount factor and ζ is the death rate.

The household subject to two budget constraint such that

$$\dot{b}_t = \left(1-\tau_t\right) w_t z_t \ell_t + r_t^b b_t + r^{borr} b_t 1_{\{b_t \leq 0\}} - d_t - \chi\left(d_t, a_t\right) - c_t + T_t$$

$$\dot{a}_t = r_t^a a_t + d_t$$

$$b_t \geq -\underline{b}, \quad a_t \geq 0$$

where z denotes the different people earns different income. c_t is the consumption of each people. a_t is the asset that they hold to invest and hedge their income risk. w_t is the income or endowment they earn which can be observed at the beginning of each instant period. $r_t^a$is the

[4]There is indeed some different no matter in the perspective of wage rigidity problem discussed by Boppart et al. (2018)or MPL problem discussed by Auclert and Rognlie (2018)

capital gain or asset gain they received because of holding a_t amount of asset. l_t is the labor supply of the household that they provide to the firm for production. τ_t is the wage income tax and T_t is the lump tax transfer which the government and central bank can use to implement the conventional monetary policy. b_t is the amount that the household deposit in the bank. Respectively r_t^b is the interest rate that the bank pay to the household who deposit in and the r^{borr} is the interest spreed that the borrower should additionally pay to the central bank. d_t is the amount of asset that household choose to investment to the financial institution each period and respectively $\chi(d_t, a_t)$ is the transaction cost.

2.2 Financial institution

We assume that there are infinite numbers of financial institutions living in the financial market who borrow from the household to finance their demand for bonds with different maturity. They will package their asset, the bonds with different maturity they hold, as the financial products, and sell them to the household. Each instance period, they will get the interest rate from the bonds they hold and pay the relative investment return to the household. They will maximize their expected increase of net worth (or equity) each instance period. At the same time, they will also have some preference for the long-term bonds to fulfill their demand in the long run as discussed by Vayanos and Vila (2009), who call them the "preferred habit investor". In our setting, we just assume they will bear a "holding cost" or "management cost" as when they hold more amount of short-term asset they may need hire more people to focus on the price change of this bonds or pay much attention to the transaction with others which will give them a higher holding cost.[5]

$$\max_{b_{t,\tau}, W_t^F} E_t dW_t$$

$$\text{s.t. } dW_t = -W_t^H r_t^a dt - W_t^F r_t^b dt + \int_1^T b_{t,\tau} \frac{dp_{t,\tau}}{p_{t,\tau}} d\tau - \left(\int_1^T \omega_\tau b_{t,\tau}^{\beta_{t,\tau}} d\tau - b_{t,0} + b_{t,T+1} \right) dt \tag{1}$$

$$W_t = \int_1^T b_{t,\tau} d\tau - W_t^H - W_t^F \tag{2}$$

$$W_t^F \leqslant \theta^p \int_1^T b_{t,\tau} d\tau \tag{3}$$

where W_t is the equity that the financial institutions hold. Within every instant period the arbitrageurs will choose optimal level of $b_{t,\tau}$amount of τ bonds that they want to hold, with the dollar value in present[6], taking the cost and revenue rate r_t^a and $\frac{dP_{t,\tau}}{P_{t,\tau}}$ where the $p_{t,\tau}$is the price

[5] In appendix we can prove that this setting is just equivalent to the setting in the Vayanos and Vila (2009)

[6] Alternatively we can set the face value or future value as the bonds amount the financial institutions hold. In that condition the asset value should be $\int_1^T b_{t,\tau} p_{t,\tau} d\tau$ which will embed the price term into the budget constraint as well as the balance sheet. However, that will make it harder to calculate the variance of the change on equity, dW_t.

of the bonds $b_{t,\tau}$. W_t^H is the amount of financial products that sell to the household. W_t^F is the amount of money borrow from the financial bank with the risk free rate r_t^b, which is the same as the rate that household can get when depositing in the financial bank[7]. $b_{t,0}$ is the face value that investor can get when the bond expires. Respectively $b_{t,T+1}$ is the amount of money that the investor invest to the bonds issuer, capital producers.

Proposition 1. *Assuming the affine function of the bond's price determinacy*

$$p(t, \boldsymbol{y}) = e^{-\left\langle \boldsymbol{y}_{t,\tau}, \boldsymbol{A}(\tau(t)) \right\rangle - C(\tau(t))}$$

where $\boldsymbol{A}(\tau(t))$ and $C(\tau(t))$ represent the inner connection of different bonds (i.e. an increase of the price of 10 years bond will pass this increase to other bonds with different maturate) and $\boldsymbol{y}_{t,\tau}$ is the exogenous shock following $d\boldsymbol{y}_{t,\tau} = -\left(\boldsymbol{\Gamma}(\boldsymbol{y}_t - \boldsymbol{y}^{ss}) + \boldsymbol{y}'_{\boldsymbol{\tau}}\right) dt + \boldsymbol{S} d\boldsymbol{B}_t$. $\left\langle \boldsymbol{y}_{t,\tau}, \boldsymbol{A}(\tau(t)) \right\rangle$ projects the shock term onto the domain of the price.

Then the demand of the financial institution is

$$b_{t,\tau} = \left(\frac{\varphi_{t,\tau} - r_t^a(1-\theta^p) - r\theta^p}{\omega_{t,\tau}\beta_{t,\tau}} \right)^{\frac{1}{\beta_{t,\tau}-1}}$$

Relatively the capital gain is decided by the optimal arbitrage rule that

$$r_t^a = \left(\varphi_{t,\tau} - \omega_\tau \beta_{t,\tau} b_{t,\tau}^{\beta_{t,\tau}-1} + \lambda_t \theta^p \right) \tag{4}$$

where $\varphi_{t,\tau} = \boldsymbol{y}^T \boldsymbol{A}'(\tau) + C'(\tau) + 2\boldsymbol{y}'_{\boldsymbol{\tau}} \boldsymbol{A}(\tau) + \left[\boldsymbol{\Gamma}(\boldsymbol{y}_t - \boldsymbol{y}^{ss})\right] \boldsymbol{A}(\tau) + \frac{1}{2} Tr\left[\boldsymbol{\Sigma} \boldsymbol{A}(\tau) \boldsymbol{A}(\tau)^T\right]$ and $\boldsymbol{\Sigma} = \boldsymbol{S}\boldsymbol{S}^T$. λ_t is the Langrange multiplier of the macroprudentical policy constraint or leverage ratio constraint.

From equation 4 we can easily find that there is a positive correlation between the points in yield curve$\boldsymbol{A}(\tau)$ and the capital gain in this period earned by household, r_t^a. In other words, assuming a decrease of yield of long term bonds which acts as a reducing $\boldsymbol{A}(\tau)$, the investment return will also decrease and the extent of decrease will be decided by 4. As we have discussed before, QE poicy that imposed by America will decrease $\boldsymbol{A}(\tau)$ through Large-Scale Asset Purchases that exposes the balance sheet of the FED. For simplicity we just assume the FED spend all of its money buy back the treasure bonds which will not influence the supply or demand side of corporate bonds market directly but indirectly through decreasing the yield curve. Since we model the yield curve and long-term interest rate endogenously, we can assume that the FED's impact on the yield curve is equivalent to a jump in $\omega_{t,\tau}$ which can easily see positively correlated with yield $A(\tau)$ and negatively correlated with bonds demand $b_{t,\tau}$.

[7]We should fulfill the condition such that $r_t^{borr} > r_t^a$, otherwise all of the household will choose to borrow and invest in the financial institution. Thus we cannot let the financial intermediate borrows from the financial bank via the interest rate r_t^{borr} which will result in no borrowing and no liquidity constraint.

However the proposition 1 absolutely cannot decide either$b_{t,0}$ or $b_{t,T+1}$and unfortunately we cannot decide their value neither in the supply side. Thus we need some assumption to pin down their exact value and i use the assumption $b_{t,0} = b_{t,1}$ and $b_{t,T} = b_{t,T+1}$ here to pin down their value. This assumption is indeed reasonable and meaningful since the maturity in our setting is continuous. There is a thin time interval for the investor to both decide the change of bonds amount to expire and finish this final transformation. Therefore they will only choose to sustain the bonds level to next period to expire and buy a new long term bonds which is the same amount as the longest existed one.

Consider the QE's first-order effect on the bonds market. Given the supply side unaffected, the QE will decrease the long-term yield first which will in turn decrease the expected income of the financial institutions. Then the financial institutions will decrease the demand of issuing financial product and the interest return given to the household. This is just the same as a change in $y_{t,\tau}$ and $\omega_\tau$that will increase the rate r_t^aas in equation 4. However, all of this analysis is based on the zero bonds supply, or in other words fixed bonds supply, which is widely used, as well as in Ray et al. (2019). In fact it may cause a faulty result as we discuss in corollary below.

Corollary 1. *Assume the long-term supply is $B_{t,\tau}$ in time t satisfying the market cleaning condition $b_{t,\tau} = B_{t,\tau}$. The optimal arbitrage rule modified as*

$$r_t^a = \left(\varphi_{t,\tau} - \omega_\tau \beta_{t,\tau} B_{t,\tau}^{\beta_{t,\tau}-1} + \lambda_t \theta^p\right) \tag{5}$$

Any $B_{t,\tau} > 0$ will ameliorate the positive correlation between long-term yield rate and short-term investment return of household. Meanwhile there exits a $\bar{B}_{t,\bar{\tau}}$ where $\tau < \bar{\tau}$ such that whenever $B_{t,\bar{\tau}} > \bar{B}_{t,\bar{\tau}}$ the correlation between long-term yield rate and short-term investment return will go to negative. In other words, a decrease of long-term rate (may caused by the QE policy) will increase the investment return rate of household.

If the supply is not zero, as we assumed previously, the GE effect can mitigate the PE effect or even override it as they comes from inverted direction where one increase the bonds bonds and one decrease the bonds. If the GE effect is powerful enough and the firm wants to issue less amount of short-term bonds instead of long-term bonds, the financial institutions will decide to borrow more instead of decreasing the liability which is supported by our quantitative result. In some sense corollary 1 is another contribution of our paper which challenges the traditional too-simple assumption, zero supply, in New-Keynesian model and shields the light on the importance of bonds supply side.

There is another interest result related to the optimal choice between the foreign bonds and domestic bonds. The arbitrageurs bear two interest cost, r_t^a and r_t^b. The arbitrageurs should balance their cost between these two type of bonds and that balanced is summarized that

Lemma 1. *The financial institution will choose to sell their financial products and set the return*

following the rule that

$$\lambda_t = r_t^a - r_t^b$$

where λ_t is the Lagrangian multiplier which satisfied

$$\lambda_t \left(\theta^p \int_1^T b_{t,\tau} d\tau - W_t^F \right) = 0$$

It is easy to see only if the interest rate was the same the constraint could not bind. While in reality at most of the time the relationship between these two rate is $r_t^a > r_t^b$. The cost of borrowing foreign is smaller thus the arbitrageurs will borrow as much as they could and reach the constraint. Additionally as the international interest rate goes lower the constraint becomes tighten which means the multiplier λ_t becomes larger. That can also be called the risk premium which is as same as Wu and Zhang (2019b) described in equation 3.5 while they got this result by no-reasonable conjecture and we provide the detailed endogenously mechanism.

2.3 Arbitrageurs

Not only the preferred habit investor, but also the arbitrageurs who can smooth the yield curve and make the financial market much more efficient lives in the economy. Similar to the setting of financial intermediates, the arbitrageurs will also choose how much of short term bonds and long term bonds to hold, and they will earn the bonds return in each period as well. We call them 'arbitrageurs' since they can also invest in risk free bonds (equivalent to the household that can deposit their money in the financial bank) to get the risk free return, which is r_t^bin our model. By endogenously choosing the amount of bonds to hold, the arbitrageurs fill the gap between corporate bonds market and money market through the choice to deposit in financial banks. They do not only increase the efficiency of our financial market to make it closer to the real world, but also provide some extent of liquidity to the producer to fulfill their increased propensity to supply more debt to financial market for investment after QE is implemented. Moreover, in our model, it is their existence that can amplify the supply side effect of QE enough to make sure the increase of output in the end, which is just the goal of the QE that take the economy out of recession. Otherwise, there may exist an equilibrium, as the opposite condition described in Corollary 1, which sustains an demand-draw drop of capital which exacerbates the recession and may financially take it into the stagnation trap or secular stagnation. In some sense, this is another contribution of our paper, which provides a situation that QE may not always be good for an economy, and its power will depend on the efficiency or friction of an economic entity. Or we can say the power of QE, in some sense, act similarly to another unconventional monetary policy, forward guidance, in the dimension of completeness of the financial market. This conjecture emphasizes that a small and developing may not directly follow the United States' unconventional monetary policy, which can only stand on the stone of the complete financial market.

We assume the arbitrageurs want to maximize their equity by solving following problem

$$\underset{b^a_{t,\tau}}{max} \boldsymbol{E}_t dW^A_t - \frac{\varphi}{2}\text{Var}_t dW^A_t$$

$$\text{s.t. } dW^A_t = \left(W^A_t - \int_1^T b^a_{t,\tau} d\tau\right) r^b_t dt + \int_1^T b^a_{t,\tau} \frac{dp_{t,\tau}}{p_{t,\tau}} d\tau + \left(b^a_{t,0} - b^a_{t,T+1}\right) dt \tag{6}$$

where W^A_t is the equity value of the arbitrageurs, $b^a_{t,\tau}$ is the present bonds value that arbitrageurs hold in time t with τ time left. $p_{t,\tau}$ is the bonds price and relatively $\frac{dp_{t,\tau}}{p_{t,\tau}}$ is the return of holding bonds $b^a_{t,\tau}$ in time t. φ is the risk averse of the arbitrageurs and that is one of the metric that measure the extent of completeness of the financial market. When φ is higher, which means the arbitrageurs become more cautious about the bonds market, the completeness of the financial market is shown decreased. Yet this is only one dimension that describe the completeness of the financial market, the existiness of arbitrageous in the market is also a dimension of completeness. If there is no arbitrageurs existing, the supply will all be consumed by the financial intermediate which means $\int_1^T B_{t,\tau} d\tau = \int_1^T b_{t,\tau} d\tau$. This will in fact cause an useless of the QE policy and we will talk about it in next section.

From the optimal condition of dW_t and VardW_t we can know that there is an indeterminacy problem when we solve the demand function of $b^a_{t,\tau}$ as there only order 1 in budget constraint. Thus we need to modify the budget constraint equation 6 to

$$dW^A_t = \left(W^A_t - \int_1^T b^a_{t,\tau} d\tau\right) r^b_t dt + \int_1^T b^a_{t,\tau} \frac{dp_{t,\tau}}{p_{t,\tau}} d\tau + \left(b^a_{t,0} - b^a_{t,T+1}\right) dt - \zeta\left(b^a_{t,\tau}\right)$$

where $\zeta(\cdot)$ is a convex function and we can interpreter it as the management cost or holding cost. For simplicity we directly assume the form of it is $\zeta(\cdot) = \iota \int_1^T \left(b^a_{t,\tau}\right)^\psi$ and the indeterminacy problem is solved by it.

2.4 Capital producer

There is a bunch of capital produce who produce the capital each period and rent them to the firm for production. Separating the intertemporal optimization problem of investment from the firm and assuming it is the producer who issues the corporate bonds with different maturity will simplify the model without losing anything, as proved by Carceles-Poveda and Coen-Pirani (2010). They will solve the maximum problem such that maximize their expected net worth subject to the budget constraint.

$$\max_{b_{t,\tau}, I_t} E_t \text{d}W^I_t$$

$$
\begin{aligned}
\text{s.t. } dW_t^I &= r_t^k K_t dt - \int_1^T B_{t,\tau} \frac{dp_{t,\tau}}{p_{t,\tau}} d\tau - \int_1^T B_{t,\tau}^a \frac{dp_{t,\tau}}{p_{t,\tau}} d\tau - p_t^K I_t dt - \phi(I_t, K_t) dt \\
&\quad + \left(b_{t,T+1} - b_{t,0} + b_{t,T+1}^a - b_{t,0}^a\right) dt \\
P_t^K K_t &= W_t^I + \int_1^T B_{t,\tau} p_{t,\tau} d\tau + \int_1^T B_{t,\tau}^a p_{t,\tau} d\tau \\
dK_t &= \left(I_t - \delta K_t\right) dt
\end{aligned} \tag{7}
$$

where W_t^I is the net wealth of the capital producer. K_t is the capital that capital producer rents to the intermediate firms. r_t^k is the real rental rate that the firms pay back to the capital producer for borrowing the capital and δ is the depreciation rate that the proportion of the capital will depreciate each period. $I_t$is the investment in capital each period with relative real price p_t^K which will also cause some amount of investment cost described by a convex investment cost function $\phi(I_t, K_t)$. The capital producer should pay the bonds interest rate each period and the total amount of the corporate bonds is the liability side of the capital producer.

For simplicity we just assume that the capital production market is totally competitive, which means that the price of the capital is just the same as the investment (and the final goods). Each producer will not hold any equity and profit each period which states the condition such that

$$
K_t = \int_1^T B_{t,\tau} d\tau + \int_1^T B_{t,\tau}^a d\tau \tag{8}
$$

$$
\begin{aligned}
r_t^k K_t dt - \int_1^\infty b_{t,\tau} \frac{dp_{t,\tau}}{p_{t,\tau}} d\tau - \int_1^\infty b_{t,\tau}^a \frac{dp_{t,\tau}}{p_{t,\tau}} d\tau + (b_{t,T+1} - b_{t,0} \\
+ b_{t,T+1}^a - b_{t,0}^a) dt - I_t dt - \phi(I_t, K_t) dt = 0
\end{aligned} \tag{9}
$$

The investment will totally be decided by the demand side, firm.

Therefore we can see the capital producer passes the risk of the bonds' price fluctuation to the financial intermediaries since it does not have any equity and use up all its cash flow to pay the bonds interest rate. The financial intermediaries will bear the financial risk that comes from the capital securitization and provide the risk-free packaged financial products to the household. Meanwhile, the financial intermediaries will also bear a liquidity problem since they cannot borrow as much as they want from the financial bank with a risk-free rate. Intuitively we can predict that the basic income of lending capital composes the investment return earned by household, minis a risk premium since the financial intermediaries endures the bonds price fluctuation, plus a liquidity premium since the financial intermediaries can also borrow from the financial bank with the risk free rate while is bound with borrowing constraint. Combining this with some assumption, we can drive the asset return of the household as follows.

Proposition 2. *Assuming for and t and τ, the condition $\beta_{t,\tau} = 1$and no holding cost such that $\omega_{t,\tau} = 0$ will always exist. In term of this assumption we can decompose the asset return that the household get as*

$$r_t^a = r_t^k - \delta - \underbrace{\left| \frac{cov(r_t^k, \frac{dp_{t,\tau}}{p_{t,\tau}})}{E_t\left(\frac{dp_{t,\tau}}{p_{t,\tau}}\right)} \right|}_{\text{risk premium}} + \underbrace{\lambda_t \theta^p}_{\text{liquidity premium}} \tag{10}$$

where $cov(r_t^k, p_{t,\tau})$ is the risk premium and a negative value, $\lambda_t\theta$ is the liquidity premium and a non-negative value.

3 Firm

3.0.1 Final good producer

Final good producer aggregate all intermediate input which produced by intermediate goods producers massed 1 to product the final good with the production that

$$Y_t = \left(\int_0^1 y_{j,t}^{\frac{\varepsilon-1}{\ell}} dj \right)^{\frac{\varepsilon}{\varepsilon-1}}$$

where $\epsilon > 0$ is the elasticity of substitution across goods.

3.0.2 Intermediate goods producers

The intermediate goods producers i use the standard Cobb-Douglas production function

$$y_{i,t} = A_t k_t^{\alpha} l_t^{1-\alpha}$$

to product the intermediate goods i suffering wage w_t. $y_{i,t}$ is the amount that intermediate goods producer i produces, using the capital k_t and labor l_t. $A_t$is the total factor productivity.

They set the price based on the Rotemberg (1982) method that

$$\underset{p_t}{Max} \int_0^{\infty} e^{-\int_0^t r_s^a ds} \left\{ \tilde{\Pi}_t\left(p_t\right) - \Theta_t\left(\frac{\dot{p}_t}{p_t}\right) \right\} dt$$

where $\tilde{\Pi}_t\left(p_t\right) = \left(\frac{p_t}{P_t} - m_t\right)\left(\frac{p_t}{P_t}\right)^{-\varepsilon} Y_t$ and $\Theta_t\left(\frac{\dot{p}_t}{p_t}\right)$ is the cost function of setting price such that $\Theta_t\left(\frac{\dot{p}_t}{p_t}\right) = \frac{\theta}{2}\left(\frac{\dot{p}_t}{p_t}\right)^2 Y_t$[8]

[8]From the FOC of household problem we can know that the stochastic discount factor $e^{-\rho\Delta}\frac{u_{t+\Delta}}{u_t}$ is just the same as $e^{-r_t^a\Delta}$

By solving the production maximizing problem we can get the Phillips curve such that

$$\pi_t = \frac{\varepsilon}{\theta} \int_t^{\infty} e^{-\int_t^s r_\tau^a d\tau} \frac{Y_s}{Y_t} (m_s - m^*)\, ds \tag{11}$$

3.1 Government and central bank

The government and central bank is the supplier of the deposit of the household. They use the tax income from the household to sustain the balance sheet themselves. The budget constraint of the government is that

$$r_t^b B_t^h - r_t^b W_t^F - r_t^{borr} B_t^{h,b} + \left(W_t^A - \int_1^T b_{t,\tau}^a d\tau \right) r_t^b dt = \tau w_t L_t - G_t - T_t \tag{12}$$

where B_t^h is the amount that the household deposit in the financial bank and $B_t^{h,b}$is the amount that the household choose to borrow from the financial bank. G_t is the government expenditure.

They will also implement a monetary policy following a Taylor rule such that

$$r_t^b = \bar{r}^b + \Phi \pi_t + \epsilon_t \tag{13}$$

However, the normal interest rate cannot go down below its nature limit, zero as the implied constraint $r_t^b = max(\bar{r}^b + \Phi\pi_t + \epsilon_t, 0)$.

3.2 Equilibrium

Summing together we can define the equilibrium of the economy that

Definition 1. There exist a path of $\left\{a_t, b_t, c_t, d_t, l_t, n_t, k_t, r_t^a, r_t^{borr}, r_t^b, w_t, r_t^k, p_{t,\tau}, b_{t,\tau}, \lambda_t, \tau_t, T_t, G_t\right\}_{t\geq 0}$ and the joint distribution $g_t(a_,, b, z)$ that satisfy the market clearing condition

$$\begin{aligned}
b_{t,\tau} &= B_{t,\tau} \\
L_t &= \int\int\int z_t l_t(a,b,z) g_t(a,b,z) dadbdz \\
B_t^h &= \int\int\int b_t(a,b,z) g_t(a,b,z) dadbdz \\
W_t^H &= \int\int\int a_t(a,b,z) g_t(a,b,z) dadbdz
\end{aligned}$$

As well as the equation 2, 8 and 12.

4 Stationary numerical solution

After we set up the baseline model we can match the model to the data and numerically solve it to see its power that explains the real world. This section mostly talks about the quantitative result of our baseline model mainly contains how to solve it, how to match the data to the model and the result.

This section firstly extends the baseline a little that introduce the equity market of the firm and uses the basic Lucas tree model to modify that market following Kaplan et al. (2018). This thin modification will make the model closer to reality as the equity boom is also an important result of the QE, which has been observed by the stock market. Then we describe the detail of the calibration process, which links our model to the real data. After describing the calibration step, we briefly explain the method that we use to solve this heterogeneous agent model, which is indeed complicated. When we finish the method part, the main part of this section and supports our previous intuition introduced in the first section. Finally i provide some robust check to further verify our model's power that explains the mechanism between QE and inequality.

4.1 Stock market

Following Kaplan et al. (2018), we assume that the capital producer can also hold the share of the firm and the amount of share is normalized to 1. Therefore the balance sheet of the capital producer, equation 8 can be modified to

$$K_t + q_t = \int_0^\infty B_{t,\tau} p_{t,\tau} d\tau$$

where q_t is the total equity value of the firm. Meanwhile the firm will provide dividend to the share holder, capital producer with a constant ratio of the total profit which we donate ω. The rest of the profit will be distribute out to the worker proportionally to their labor productivity z_t. We can regard this setting as the bones based on the profit or the employee stock ownership which is really popular in US. Based on the principle introduced in Lucas Jr (1978), we assume their are also a lot of arbitrageurs in the stock market who will financial pin the total equity value to the discounted dividend such that

$$q_t = \frac{\omega \Pi_t}{r_t^k - \delta}$$

After the equity price is endogenously decided we can embed the stock market into our model.

4.2 Calibration

Most of the parameters are estimated matching the data of US in 2008, in which the financial crisis started. We will explain the calibration method and process sector by sector from the household to the government. At the end we summery these parameters into the tabel.

4.2.1 Household

We assume that the utility function of the household is separable CRRA utility function follows the form

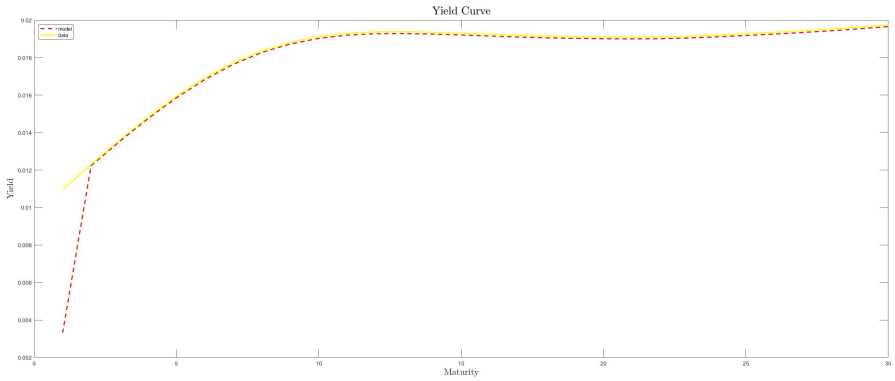
$$u(c_t, l_t) = \frac{c_t^{1-\sigma}}{1-\sigma} - \kappa \frac{l_t^{1+\nu}}{1+\nu}$$

where $\frac{1}{\sigma}$is the intertemporal elasticity of substitution. κ is the labor disutility coefficient. $\frac{1}{\nu}$ is the Frisch elasticity of labor supply. For simplicity we just assume $\sigma = 1$ so that the utility of consumption is log form. Similarly we assume $\nu = 1$ which is also widely used in literature. We calibrate the κ so that the average hours supplied by the household is $\frac{1}{2}$ where $\kappa = 2.2$. The deposit rate provided by the financial bank is setted at 0.0057 which is the Federal Funds Rate in November of 2008. The interest rate that household need to pay if they borrow money from the financial bank, as well as the parameters used in the transaction cost of investment are calibrated by Kaplan et al. (2018) to match the wealth distribution in SCF. We slightly change the borrowing rate since when the recession comes each financial bank will be face of the cash constraint and increase the borrowing rate. We set the death rate ζ to $\frac{1}{180}$ which means that the expected working time of a household is 45 years.

4.2.2 Financial institution

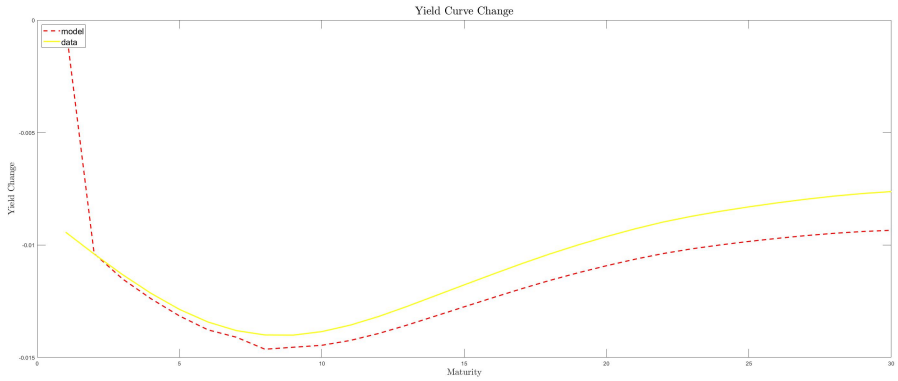
There is really a lot of parameters we need to calibrate. Firstly we target the capital-output ratio in 2008 as 3.7 and i3.45 in 2010 which are also supported by Woo and Kumar (2015) and Werning (2015); Gourio and Klier (2015). Since the prudential policy and the liquidity premium is not what we want to focus on in this paper, to make sure the binding collateral constraint for simplicity, we just assume the constraint parameter $\theta = 1\%$. Later we will reconcern it and prove that this setting will not influence our result too much at the robust check section. r_t^a and r_t^b can both be endogenously decided by our model. However since the problem we want to analyze is the QE policy which is proposed when the central bank was facing the ZBL, we will set r_t^b to 0.57% which is the Federal Funds Rate in November of 2008. Then we calibrate the y function with respect to the τ to make the yield curve match The Treasury High Quality Market(HQM) corporate bond yield curve in November of 2008, setting the variance σ_τ as 0.0116 which is calibrated by Ray et al. (2019). Together with the calibration of the left parameters ω_τ and β_τ to target the data for minimizing the sum of distance. Figure1 shows the difference between the data and our model after calibration.

Figure 1: Yield curve between data and model



Similarly we set the change of y with respect to τ after the QE in March of 2010 assuming this change is permanent and the economy jump to a new stationary equilibrium which shown in Figure2. Yet we can only target this change partially because of the time limit of our code, we cannot calculate the general equilibrium result so fast that can be used to calibrate. Thus we firstly calibrate the partial effect of y's change caused by QE to match the yield curve, later adjust the ω_τ based on the function provide in Ray et al. (2019) which is the resource of the real effect of the QE on the economy. In next section we will discuss the change detailedly.

Figure 2: Change of Yield Curve



4.2.3 Arbitrageurs

There are three parameters in this section that we need to calibrate and other left are all the same that are decided by the bonds market. The risk aversion φ of the arbitrageurs is fixed to be 1555 so that there is an appropriate bonds holding level. Besides, this value is also closed to the risk aversion calibrated by Ray et al. (2019). We in turn set $\psi = 2$ so that in FOC there is only order 1 in that equation. Because we set the holding cost part $\zeta(\cdot)$ aiming at solve the indeterminacy problem of the arbitrageurs, the scalar ι should be smaller enough to make sure there is no huge effect on the decision making of the arbitrageurs. Therefore we set that equal to 0.01 for simplicity.

4.2.4 Capital producer

To the purpose of simplicity, we further reduce the complexity of the model and assume that the investment cost is zero so that equation 9 becomes $(r_t^k - \delta)K_t dt + \omega\Pi_t - \int_0^\infty b_{t,\tau}\frac{dp_{t,\tau}}{p_{t,\tau}}d\tau = 0$. We set ω to 30% which is support by the NIPA table 2.1 and 5.1. Additionally we set the depreciation rate δ to 6% annually which is a reasonable data in literature.

4.2.5 Goods producer

All of the parameters that are used in the financial goods producer and intermediate goods producer are standard and widely used. We summary these parameters in the Table 1

Table 1: List of parameters in production sector

Parameter	Description	Value	Source
ϵ	Elasticity of final good producer' demand to intermediate goods	10	Steady-state markup $m^* = 11$
θ	Rotemberg price adjust cost	100	Schorfheide (2008)
α	Capital share of Production in Steady state	0.33	

4.2.6 Government and central bank

Since the conventional monetary policy is no longer applicable when the short-term risk free interest rate touch the ZLB, the canonical monetary policy, Taylor rule, or equation 13 cannot be successfully implement. Therefore the parameters that govern the interest rate setting such as $\bar{r}^b$ and ϕ is insignificant in our model and we do not need to consider their calibration. The only rest of the parameters is the tax which we set to be 20% which is the mean tax level in US.

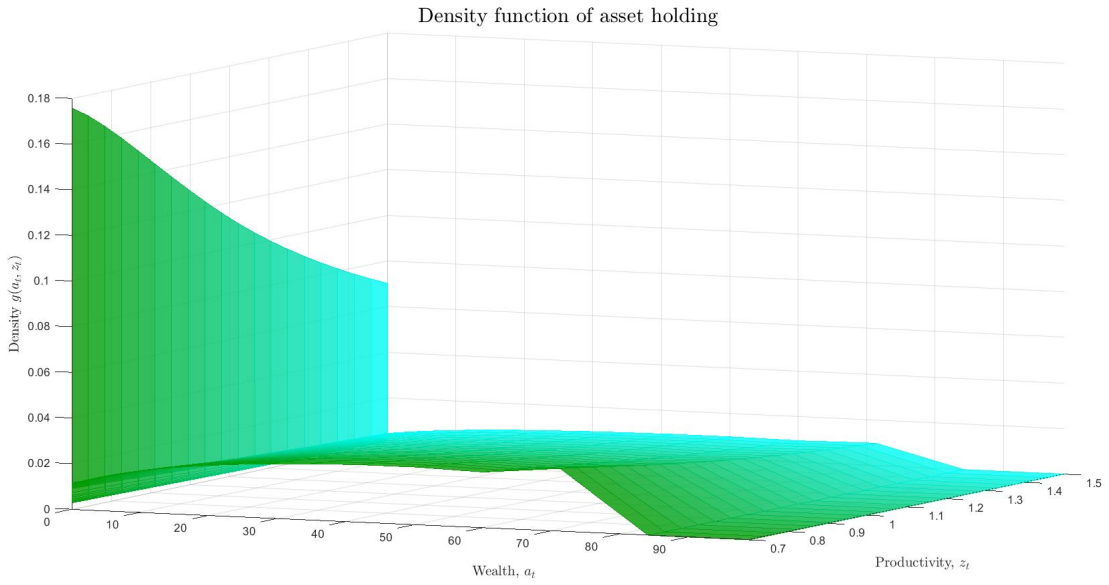
4.3 Quantitative result in stationary equilibrium

4.3.1 Wealth distribution in stationary equilibrium

At the stationary equilibrium standing at the beginning of the quantitative easing policy, November of 2008. The yield curve of data, as well as that predicted by the model, is already shown in Figure 1 which indicates the mighty power of our model to explain the real world. In our model, all of the households can either invest in the financial intermediates to get the profit from the financial products, such as the funds, financial insurance, and fixed-term deposit in reality. However since there exists the transaction cost when the households want to invest or get back their principal from the financial institutions, each household will not invest all of their asset or wealth into the financial market as they also need to smooth their consumption through the timeline. Therefore we can predict that only part of the household will invest into the financial intermediates to get a higher return and a bunch of households will not hold any financial product since they are poor enough to squeeze out the money and labor income to investment. They are

eager enough to consume now since their consumption level is low and they prefer to spending all of their income to consume instead of saving or investing. They may be hungry even further and choose to borrow to consume. The model fully describes these in Figure 3. We can observe from this Figure that there is a lot of household crowd at the zero asset holding point as there contains the largest density of the asset distribution. The density becomes flat, not increasing or decreasing, along with the increased households' wealth. It is worth to notice that there is a trend of increasing density at the low level of wealth holding since it is beneath them to invest in financial intermediates as there exist the transaction cost and the density goes flat later. At the far end of the wealth, the density decreases stating that there are less and less of households holding relatively large amounts of assets. In other words, there are a handful of people hold plenty amount of assets, and there is inequality problem in society.

Figure 3: Distribution of asset



In addition to analyzing along the a_t direction, we can also focus on the productivity direction, z_t direction. We can see as productivity increase, the density of distribution is decreasing. It is intuitive since following a larger productivity, the household earns more and more wage income. They do not need to invest in financial intermediate to get return to fulfill their consumption as they already get enough money by working in each time period. They can directly use their wage income to consume instead of consuming by investing. Furthermore, the descent density along the productivity is not the same on all levels of asset holding. In fact it is decreasing as the asset holding increasing. When you hold less assets on hand, for instance, no asset comparing to plentiful assets, you will be more willing to invest your high wage since all of the households are suffering from uncovered idiosyncratic income shock. Therefore there will be fewer household choose not to hold asset when they have higher productivity. This is revealed by a larger decrease of density at the lower asset level, and increased productivity.

Though the household can invest in financial intermediates to get a high asset return back to consume, they are still fear the unstable consumption level as there are unhedged income risk and they are all risk averse. They cannot take out their investment as frequently as they want to smooth the consumption since the intermediates need them to pay the transaction cost. Fortunately, a financial institution is not the only one that they can store their income or share their risk along the time line. They can also deposit their money at the financial bank bearing a much lower risk-free interest rate. Meanwhile, for those poor and urgent demand for consumption, they can also borrow from the financial bank with a much higher interest rate. It is intuitively to think that when the majority of an economy entity is poor or hand to mouth household, there will be a crowd at the borrowing stage or zero deposit stage, as shown in Figure 4. We can see there is a lot of households' deposit gathering at the zero around, which demonstrates our initiation that most of the households are poor and middle-income people who will prefer saving less into their check account or directly use the credit card to borrow a little to meet with their short-term consumption demand. They all tend to invest in financial intermediate.

We can use the saving function, drawn in Figure 5, to investigate the household problem carefully. Figure 5 describes the saving function of different households with different income productivity (equivalent to different wage income levels). We can see that low-income households do not want to save anymore since their income level is so low that they support their consuming demand. They need to borrow money to smooth their consumption demand and paying back their debt later, where the graph shows that when there is a negative position of the household's bank account, they will choose to pay back their debt slightly. In fact this is also partly contributed by the working productivity schedule. The low-productivity household assume that they can be better, or in other words, have a higher possibility of jumping out of the low income plight as we assume the idiosyncratic shock follows the O-U process. Thus by expecting a higher income later, they would have a higher propensity to borrow more to consume today. Conversely, if we assume that the productivity shock is in some extent a poisson process in which it is much harder to climb up from the low-income level to high-income level than the opposite direction, the household with low productivity will not choose to borrow as much as they did in the baseline model. Additionally, we can see their saving function is an upward shifting comparing to the low-income productivity. However, this shifting follows a twist on the negative part where the saving function goes steeper than that of low-income productivity. This is natural since the high-income household hold plethora cash in each period. Besides fulfilling their consumption demand, they will choose to decrease their debt level as far as where they can reach. Thus as the debt level increases, their saving amount (or repaying amount) will absolutely increase.

Figure 5: Saving Function

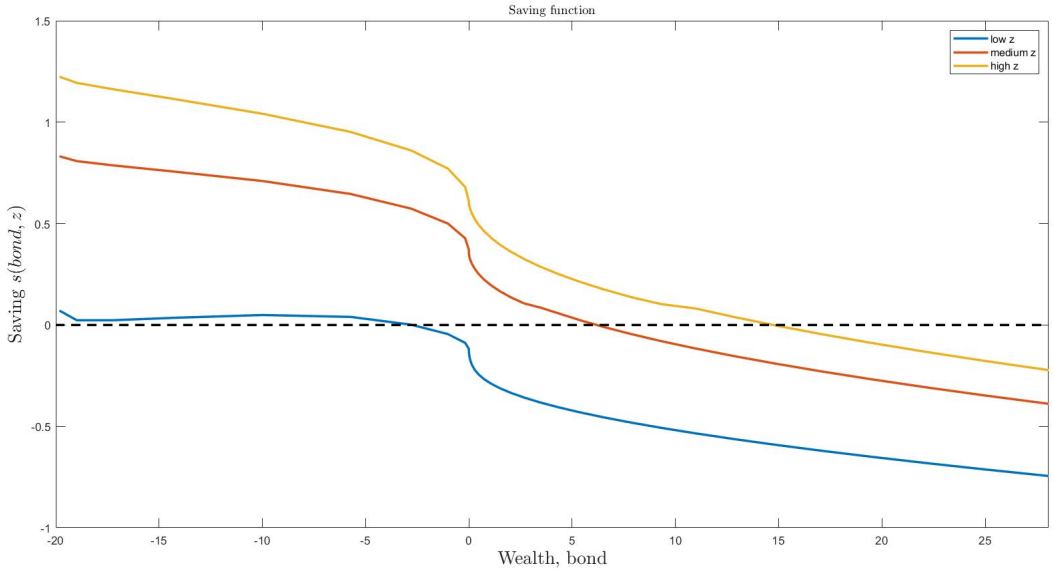
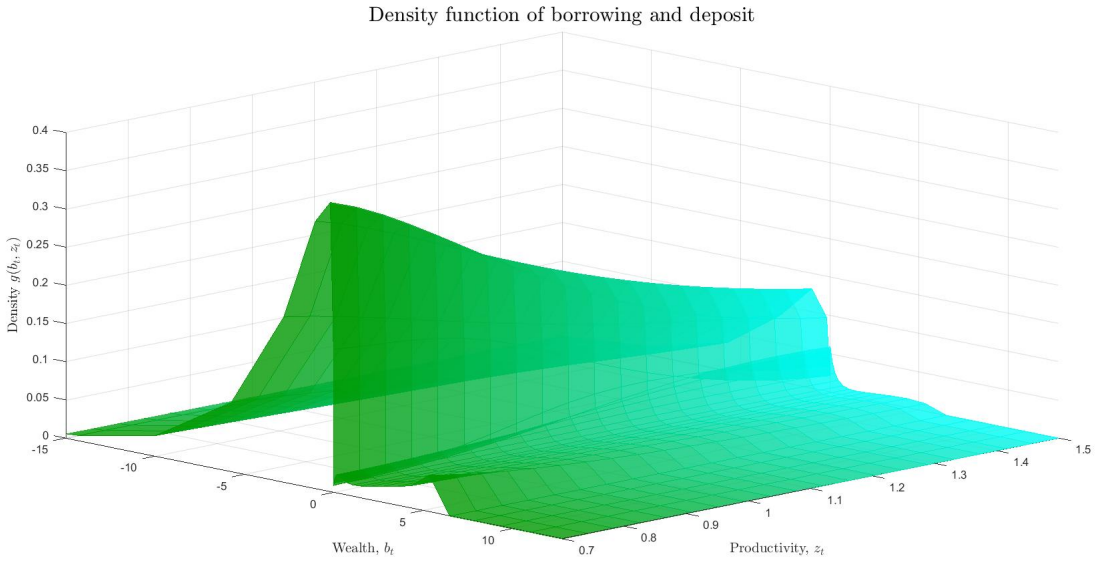


Figure 4: Distribution of deposit



4.3.2 The effect of Quantitative Easing Monetary Policy

Given the QE one when the FED expense its balance sheet by buying the long-term treasure bonds, which is just what they issued. This action, in fact, does not directly "pay" money to the market but decreases the long-term yield by using the right hand to fetch money on the left hand. However since the long-term interest rate in the market is following, other financial institutions that hold the money but are pretty cautious about the economy condition in recession will inject some liquidity into the market as they cannot secure their asset return high enough. This liquidity

will somehow save the firm and, in some extent, sustain their production. Based on our model, the decreased long-term yield will cause a decreased asset return of financial assets that sold to households. This decreased return will encourage the household to withdraw some part of their saving to consume, which in turn result in a demand-side drawing boom. From our experiment, the increased ω are so tiny that they satisfy $\max \frac{\Delta\omega_{t,\tau}}{\omega_{t,\tau}} = 3\%$ where $\tau = \{26, 27, ..., 30\}$. In other words, the maximized increase of ω is only 3% (in fact most of the percent changes are below 0.01%) while can stir up a 16.7% decrease in households' asset return. Because of this decreased return, people who hold the asset will experience a loss of welfare if they did nothing. Observing this decreased asset return, they will decrease the holding of assets and use this liquidity to consume, which previously we can easily propagandize by reducing the interest rate. Yet when the interest rate goes close to zero where the central bank can no longer reduce to, To propose household further consume to get economy back from the recession, the QE works on the asset market to influence the return of asset, rather than risk-free short-term rate. Since not all households having an account in the financial intermediates, not all of the household holding the same amount of the financial products, the effect of the QE is disproportionate. It is easy to imagine that the poor people will not be affected at all as they do not hold any money invest in the financial institutions. Only the middle income and rich class who have the asset in hand will get the impact of the QE. It is good news for the poor people since at present they are so poor that do not have any more to lose. QE policy stimulates the economy, increase their wage income[9]. Despite the influenced part, the middle class and rich people pay their money to consume to draw the economy, their endeavor is not the same, which in the end causes the inequality problem. The rich have accumulated a huge amount of wealth, which provides enough asset return for them to consume, even though they do not work or have the lowest labor productivity, in our setting. When the asset return decreases, they will only decrease the speed of accumulating wealth, or slightly decrease the asset level from the intermediates, because MPC is decreasing as the consumption level increases and households already stay at consumption level high enough. On the contrary, the middle-income people are the main part that stimulates the economy and also the impaired one. They hold relative small amounts of assets invested in intermediates. The decreased asset return irritates their decreasing account in intermediates as the expected future return is smaller, and it is worth to draw money back to consume now. In other words, the QE policy extrudes their asset position skew on the poor point and increase the chasm between the median wealth point to the rich point. In other words, QE increases the inequality problem by keeping rich guys rich again and middle-income guys becoming poor.

This is supported by our experiment, as shown in Figure 6. This is the distribution of asset people hold after the QE. Comparing Figure 3 with this figure, we can notice that the whole pattern does not change at all. Meanwhile , the rich people's density does not change either which means that the QE policy does not affect the rich people a lot, as discussed before. However,

[9]If we also consider the employment and unemployment problem, the condition will not be changed a lot, as discussed in Heathcote et al. (2020).

that of the poor people, where the far left of the diagram, increases a lot under QE's impact. It is almost increased by 11%, which is a large number and only caused by QE, a single policy. The gnin index of asset holding from our experiment increases from 0.42384 to 0.42450, a 0.2% increase. Our experiment is also supported by the data. Figure 7 shows the change in corporate equities and mutual fund shares. Between 2008 Q4 and 2010 Q1, the top 1% people's holding increases from 41.9% to 47.2%. While 99%-90% percentile decreases from 41.0% to 38.7%. 90%-50% percentile group's holding decreases form 16.1% to 13.2%. This data verifies our initiation and model and demonstrates that our model can explain part of the real world.

The financial technology applied in financial intermediates nowadays will not only facilitate the household and help them enroll in the market more easily but also expedite the lending process of banks and mitigate moral hazard problems within the lending process. More specific in our model, the application of financial technology will help the household investment in financial intermediates such that they do not need to go to outlets frequently to catch the latest financial products or talk with different brokers to find the most profitable financial products. They also do not need to stay at the outlets to read the tedious document and sign the contract in working time which may require them to ask for a leave. All of these cumbersome tasks could be finished by fingertips or mouse. This will decrease the transaction cost χ a lot, which will amplify the QE effect on the total economy and recede the inequality problem as the middle-income class pays less to get back their asset.

Figure 6: Distribution of asset after QE

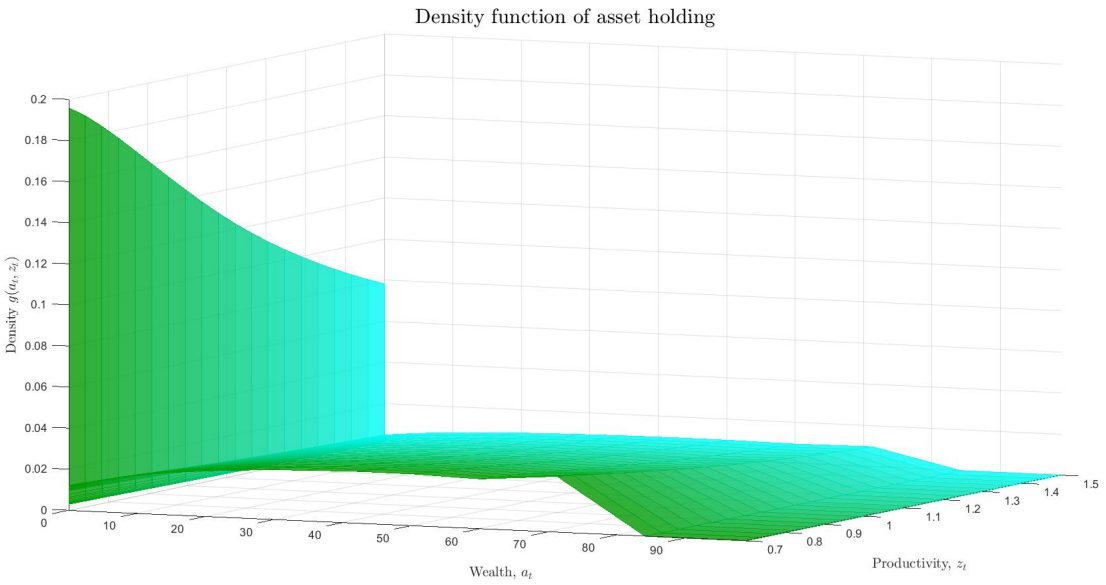
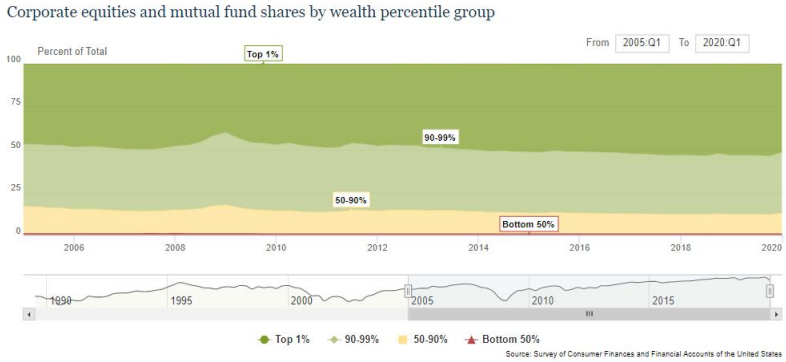


Figure 7: Wealth distribution change



It is also important to analyze the QE's impact on the aggregate variable, such as the output, consumption, capital, which are targeted at the beginning by the policy maker. The most primal thing that the policy maker considers is whether the QE could help the economy back to normal or even a new prosperity. Most of the literature agrees that QE policy works well in protecting the economy down fewer to greater recession, such as Weale and Wieladek (2016), Joyce et al. (2012), Ugai et al. (2007), and Kapetanios et al. (2012). In fact the economy recoveries back not only because of the demand-side drawing effect, but also because of the supply-side push effect. After the long-term yield curve decreasing, the firm can issue more bonds to raise money to invest and hire workers to produce. Meanwhile, the boom in equity market will increase their equity share price, which makes them easier to borrow money outside to fulfill their liquidity problem. In our experiment, we can stimulate a 1.5% increase in output, a 2.3% increase in consumption and a 5.5% increase in capital. This verifies our intuition and prove that our model can match some part of reality.

Although our paper mainly focuses on the wealth inequality problem caused by the QE policy, our model can also match and explain part of other inequality problems which is discussed extensively in the literature and among policy makers.Based on our model the income of a household is defined labor income after tax, deposit interest rate income, asset return, bonus income and lump-sum transfer, as summarized in equation 14.

$$\text{income}_i = \underbrace{(1-\tau_t)\, w_t z_t \ell_t}_{\text{labor income}} + \underbrace{r_t^b b_t + r^{borr} b_t 1_{\{b_t \le 0\}}}_{\text{deposit income}} + \underbrace{r_t^a a_t}_{\text{asset return}} + \underbrace{\frac{z_t}{\bar{z}_t}(1-\omega^b)\pi_t}_{\text{bonus}} + \underbrace{\frac{z_t}{\bar{z}_t} T_t}_{\text{lump-sum transfer}} \tag{14}$$

Table 2 shows the change in inequality and its components. We can see our model predicts that almost all components of the income's inequality become more severe except the bonus and lump-sum transfer parts which are constructed for market cleaning and negligible, as the variance of idiosyncratic does not change a lot and they take a small share of the total income. The increase in labor income inequality is reasonable as the wage increase and our model allows continuous elastic labor supply change. An increased multiplier will increase the gini index because of the definition of gini index. Deposit income inequality raises due to economic

Table 2: **Income gini index change**

Components	Percentage change of gini index
labor income	0.0335%
deposit income	0.4801%
asset return	0.7157%
bonus	$-7.9567 * 10^{-4}\%$
lump-sum transfer	$-7.9567 * 10^{-4}\%$
total income	2.2515%

recovery which increases the demand of poor and middle-income class for borrowing money from, which in fact pushes up the inequality. Overall income inequality worses.

4.3.3 Demand-side effect comparing to supply-side effect

As discussed in the previous section, QE has two effects on the economy, one is on the supply and another is on demand. QE can decrease the long-term yield, which induces a decreased demand for holding asset of households as their asset return decreased. This decreased asset return will lead to increased consumption as households do not have other channels to roll up their asset value. Meanwhile, the decreased long-term yield will also stimulate the supply side, firm, to produce more as they can issue more bonds to raise money. There is a question involuntary emerging in our mind, is the supply-side effect more important than the demand-side effect or the inverse direction? Which section in our model, as well as in the real word, is the main force that takes the economy back from recession?

We can take the experiment in mind that, assumes there is no supply-side effect and the firm cannot issue as much as new bonds as they want since there is no one in the demand side wanting to buy these new bonds. Then only the demand-side effect works to stimulate the economy. Nevertheless the heterogeneity in the demand side will mitigate the effect of the QE as only the middle-income household will do their best to spend money on consumption. The poor household has nothing to spend out anymore and the rich people spend a little more because of high enough consumption level. Comparing to the supply side, the firm will increase their capital and labor spending all, which constitutes a larger effect than that created by only part of household. Obviously, someone may retort that there is also existing heterogeneity in the firm or production section. However, the impact of heterogeneity on demand and supply side is reverse. It is the middle-income household in demand side who does the most job while their wealth and consumption level is relatively small compared to rich households. On the other hand, in bad time, it is the large companies who can more easily borrow money and issue bonds out. They are the majority of the producer that products the goods. Therefore the representative agent assumption on the production side will not change the result a lot.

To finish the experiment using our model, we need to decrease the supply-effect to see their effect on the QE policy implementation. It is worth to take notice of the financial market

Table 3: Change after shutting down supply-side effect

Variable	Description	with arbitrageurs	without arbitrageurs
r_t^a	asset return of household	-16.7%	2.38%
Y_t	aggregate output	1.5%	-3%
C_t	aggregate consumption	2.3%	-2.33%
K_t	aggregate capital level	5.5%	-8.28%
i_{gini}	gini index	0.2%	-0.15%

completeness's correlation with the supply-side effect. Even though there is a demand side stimulation at the beginning, later the financial market transfers the decreased amount of asset to the capital production side, which depresses the production. The producer's propensity to invest creates an appreciation in financial return and fulfill the depressed demand result from the general equilibrium channel. This in turn cause a severer recession. When the completeness of financial market is higher, the firm can issue bonds to raise money more easily. The financial market completeness in our model is measured by bunch of degrees such as the risk aversion extent of arbitragers, the management cost of the arbitragers and capital producers, and the bond price fluctuation variance. In our experiment, we directly assume that arbitragers do not exist in the financial market. In other words it is the most inefficient financial market we can get from our model while not losing the generality. We recalibrate the model, taking others unchanged except the depreciation rate and setting it to 4% every year.

Under the application of financial technology in the financial market, the financial intermediate could pay much less management cost during handling the intricate short and long term bonds and provide a higher asset return to the household. This will indeed aggravate the inequality problem since the asset holders now get a higher return from the financial intermediate. However, on the other hand, the financial intermediate can provide more financial assistance to the supply side. The firm would experience faster and easier financing flow, which increases production. The overall effect of financial technology is obscure and still wait for us to excavate. The experiment shows that the supply-side effect is the main part that drives the economy to the boom. When we shut down the opportunity that firms can issue more bonds to production. In fact we further assume that the firms decrease their capital as the financial intermediates reduce their bonds holding. This results in a further output loss, compared with the consumption loss, leading to a higher depression.

Our model does well not only on explaining the inequality problem caused by the quantitative easing policy, but also on matching and explaining the change in financial asset structure after the quantitative easing policy which can be considered another contribution to the literature. As discussed, the decrease in long-term yield will cause a decreased demand in financial institutions but an increased supply by capital producers. The downward shift of demand curve and upward shift of supply curve will result in a decrease in interest rate. However, whether total bond will increase or decrease depends on the general equilibrium. Economic intuition may help us to

predict the change. There is only first order effect on the demand side, financial institutions but a feedback loop on the supply side because of the household sector. Therefore the supply-side effect will dominate the demand-side effect and the total bond volume will increase. Hence, our model predicts an 10.47% increased volumeof corporate bond, which is consistent with the data.

Since yield curve is twisted, changes in volume of bond with different maturity will not be equal. Long-term bond will increase more than short-term bond. Table 4 compares changes in the share of bond with different maturity after the QE predicted by the model and those in the data. Following Giambona et al. (2020)[10]'s definition, we define the bond's share of τ as the total bonds value with maturity τ over the total bonds issued by corporate. Since they have different maturity frequency, we only compare three point there. We can see our model's prediction match the data well, which is also consistent with the intuition.

Table 4: Bonds volume share's change with different maturity caused by QE

Maturity	Debt Maturing in 2 Years	Debt Maturing in 3 Years	Debt Maturing > 4 Years
Data	-0.008	0.009	0.0287
Model	-0.0025	-0.0026	0.0398

4.3.4 Combine with macroprudential policy

In addition to the monetary policy and fiscal policy, the central bank and government can also propose a macroprudential policy to corporate with other policies to get a higher welfare distribution. There are two main types of macroprudential policy. One is focusing on the deleveraging on the household side, in other words, the demand side. Since there exists aggregate demand externality and pecuniary externality, collaborated with the asymmetric MPC within the household group, the appropriate macroprudential policy will redistribute the debt and avoid the recession which may be caused by liquidity trap, safety trap or other traps discussed in Farhi and Werning (2016), Korinek and Simsek (2016), J Caballero and Farhi (2018) and Eggertsson et al. (2019). Another one is focusing on the deleveraging of the financial intermediates where the moral hazard and incompleteness survive. The government will impose the macroprudential on the financial market to predecrease the leverage ratio of the financial intermediates to stabilize the market, which is discussed in Collard et al. (2017) and Bianchi and Mendoza (2018). Meanwhile, the leverage of financial intermediate is indeed procyclical, and in some sense it also agrees with real life. Assuming there is a macroprudential policy imposed on the financial intermediates, which causes a decrease in θ^p as the intermediates cannot borrow as much as they want from the financial bank. By our experiment, we can answer the question, what is its effect on the power of the quantitative easing monetary policy.

From our experiment, given a decreased θ^p from 1% to 0.5%, the power of QE policy on the aggregate stimulation is amplified summered in Table 5. The result shows that combining

[10]Since their latest version is focusing on the investment of corporate, they deleted related table in 2020 version. For reference people can check their 2017 version of table 7.

with macroprudential policy the quantitative easing monetary policy can stimulate the economy reaching a higher aggregate level with a lower cost no matter in interest rate or inequality. This prediction is intuitive as the macroprudential policy makes the intermediates harder to finance with a lower interest rate. Thus a decreased demand for holding long-term bonds will result in a much more proportional decreased in borrowing from the household. Because given the same amount of change in assets(long-term bonds), a lower holding of liberality in financial bank means a higher holding of household asset, facing the change in long-term bonds, the financial intermediate will persuade the household to draw more money back from them which will cause a stronger stimulation, under the macroprudential policy.

Table 5: Effect of macroprudential policy on the QE

Variable	Description	Before macpru policy	After macpru policy
r_t^a	asset return of household	-16.666%	-16.614%
Y_t	aggregate output	1.478%	1.482%
C_t	aggregate consumption	2.340%	2.344%
K_t	aggregate capital level	5.494%	5.506%
i_{gini}	gini index	0.158%	0.157%

5 Discussion and Conclusion

In this piece, we put forward the idea that the QE policy in the Unite States will make the inequality problem more several and worse through the distribution channel as different households hold different levels of wealth. We set up a general equilibrium model, together with the quantitative result derived from that model, showing that the channel indeed explains part of the increased inequality after 2008. However, there are also other channels through which QE affects the inequality extent such as equity market, MPL (employment and unemployment), and international trade. These are waiting for more research focusing on later.

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A Proof of proposition

A.1 Proof of proposition 1

Proof. Given θ^p is small enough, we can easily get that equation 3 will be binding and $W_t^F = \theta^p \int_1^T b_{t,\tau} d\tau$. Combining other constraint such that equation 1 and 2, the problem becomes

$$\max_{b_{t,\tau}, W_t^F} E_t r_t^a \left(W_t + W_t^F - \int_1^T b_{t,\tau} d\tau \right) dt - W_t^F r_t^b dt + \int_1^T b_{t,\tau} \frac{dp_{t,\tau}}{p_{t,\tau}} d\tau \tag{15}$$

$$- \int_1^T \omega_\tau b_{t,\tau}^{\beta_{t,\tau}} d\tau + b_{t,0} - b_{t,T+1} + \lambda \left(\theta^p \int_1^T b_{t,\tau} d\tau - W_t^F \right) \tag{16}$$

Solve this problem we can get the financial intermediates' demand function of bonds $b_{t,\tau}$ such that

$$b_{t,\tau} = \left[\frac{E_t \frac{dp_{t,\tau}}{p_{t,\tau}} - r_t^a (1 - \theta^p) - r_t^b \theta^p}{\omega_{t,\tau} \beta_{t,\tau}} \right]^{\frac{1}{\beta_{t,\tau} - 1}} \tag{17}$$

Equation 17 can be rewrite to the equation 4 as when

$$p(t, \boldsymbol{y}) = e^{-\left\langle \boldsymbol{y}_{t,\tau}, \boldsymbol{A}(\tau(t)) \right\rangle - C(\tau(t))}$$

we can get

$$\begin{aligned} \frac{\partial p}{\partial t} &= p_{t,\tau} \left(-\boldsymbol{y}^T \boldsymbol{A}'(\tau) \frac{d\tau}{dt} - C'(\tau) \frac{d\tau}{dt} - \frac{\partial y}{\partial \tau} \frac{d\tau}{dt} \boldsymbol{A}(\tau) \right) \\ &= p_{t,\tau} \left(\boldsymbol{y}^T \boldsymbol{A}'(\tau) + C'(\tau) + \frac{\partial y}{\partial \tau} \boldsymbol{A}(\tau) \right) \end{aligned}$$

Similarly we can also get

$$\nabla_{\boldsymbol{y}} p = -p_{t,\tau} A(\tau)$$

$$H_y p = p_{t,\tau} (A(\tau) A(\tau)^T)$$

Based on Ito's Lemma the profit of holding bonds will be

$$\frac{dp_{t,\tau}}{p_{t,\tau}} = \varphi_{t,\tau}dt - A(\tau)^T S dB_t \tag{18}$$

where $\varphi_{t,\tau} = \boldsymbol{y}^T \boldsymbol{A}'(\tau) + C'(\tau) + 2\boldsymbol{y}'_{\boldsymbol{\tau}} \boldsymbol{A}(\tau) + [\boldsymbol{\Gamma}(\boldsymbol{y}_t - \boldsymbol{y}^{ss})]\, \boldsymbol{A}(\tau) + \frac{1}{2} Tr\left[\boldsymbol{\Sigma}\boldsymbol{A}(\tau)\boldsymbol{A}(\tau)^T\right]$ and $\boldsymbol{\Sigma} = \boldsymbol{S}\boldsymbol{S}^T$

Combining equation 17 and 18 we can get the equation 4 in proposition 1. □

A.2 Proof of proposition 2

Proof. In stationary equilibrium we can assume that there is no investment cost and $I_t = \delta K_t$. Combining equation 8 and 9, and taking derivative with both side of equation with respect to $b_{t,\tau}$ we can get

$$E_t r_t^k \frac{dp_{t,\tau}}{p_{t,\tau}} = E_t \left(\frac{dp_{t,\tau}}{p_{t,\tau}}\right)^2$$

Thus

$$r_t^k = \frac{E_t \left(\frac{dp_{t,\tau}}{p_{t,\tau}}\right)^2 - cov\left(r_t^k, \frac{dp_{t,\tau}}{p_{t,\tau}}\right)}{E_t \frac{dp_{t,\tau}}{p_{t,\tau}}}$$

This can be used to derivative equation 10 □