The Market for Used Capital, Financial Frictions, and Aggregate Productivity

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Abstract

We study the role of the used capital market in generating aggregate productivity gain by mitigating firm-level distortions caused by financial frictions. Using a general equilibrium model with heterogeneous firms, market for used capital, sectoral choices, and collateral constraints, we analytically demonstrate that allowing for used capital market facilitates firms' entry into high-productivity sectors and reduce dispersion of marginal product of capital across firms within sector, which collectively reduce aggregate productivity losses at the macro level. Quantitatively, incorporating the used capital market results in a 9.3% productivity gain, with approximately 80% of this improvement arising from the extensive entry channel.

JEL Codes: D24, E22, E23, G32

Keywords: The market for used capital; Financial frictions; Collateral constraint; Aggregate productivity; Entry; Capital misallocation.

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1 Introduction

Firms trade large amounts of used capital both directly on secondary markets and indirectly through acquisitions. Among public firms in U.S, the average expenditure on used capital accounts for about 28% of total physical capital expenditures (see Eisfeldt and Rampini (2006); Lanteri (2018)). Given its importance, the implication of used capital market for aggregate productivity is largely overlooked in the literature.

Relative to new capital, used capital is cheaper upfront and thus is easier to finance. In the data, financially constrained firms, such as smaller firms and younger firms, utilize more used capital than unconstrained firms; start-ups enter the market and start their business by purchasing large amounts of used capital.[1] These evidence suggest that used capital provides firms with a way to mitigate distortions of entry decisions and investment decisions caused by financial frictions, which are shown to be an important source of aggregate productivity losses.[2]

In this paper, we study the implication of used capital market for aggregate productivity. Specifically, we investigate the novel role of the used capital market in generating aggregate productivity gain through mitigating firm-level distortions caused by financial frictions. We analytically show the role of used capital in generating positive efficiency gains through facilitating firms' entry into the high-productivity sector and reducing capital misallocation within the sector in an analytical model, and then quantity these effects in a dynamical general equilibrium model.

We develop a general equilibrium model with the market for used capital, heterogeneous firms, choices between the low-productivity traditional sector and the high-productivity modern sector, and financial frictions in the form of collateral constraints in the framework of Midrigan and Xu (2014) and Li and Xu (2023). In our model, heterogeneous producers produce the same consumption goods by operating in

[1] Using the data from the Annual Capital Expenditure Survey (ACES), Eisfeldt and Rampini (2007) show that used capital comprises 28% of capital expenditures for firms in the lowest asset decile (with assets below \$0.10 million) and 10% for firms in the highest asset decile (with assets exceeding \$186.55 million) and this fraction decreases monotonically across asset deciles. Ma, Murfin, and Pratt (2022) document that start-ups and young firms purchase a disproportionate share of old physical capital using capital transaction level data from Equipment Data Associates (EDA).

[2] Financial frictions can reduce aggregate productivity via two channels. On the one hand, they may distort entry and technology adoption decisions (the extensive margin) and thus reduce the productivity of individual producers (see Cole, Greenwood, and Sanchez (2016)). On the other hand, financial frictions may generate differences in the returns to capital across individual producers, and thus efficiency losses due to misallocation (the intensive margin, see David and Venkateswaran (2019), Restuccia and Rogerson (2017), Hsieh and Klenow (2009), and Castro, Clementi, and MacDonald (2009) among others). Midrigan and Xu (2014) is document that the extensive margin is quantitatively more important.

either the traditional sector or mordern sector. Producers in the unproductive, *traditional* sector, only have access to an unproductive production technology. They only use labor and do not require financing. Producers in the productive, *modern* sector, can produce with a more productive technology, which requires both capital and labor as inputs. To enter the modern sector, producers need to pay a fixed cost first and invest in capital for production. Producers are allowed to borrow from households to pay for the fixed cost and capital investment, but external financing is subject to collateral constraints. Following Rampini (2019) and Lanteri and Rampini (2023), the collateral constraint requires that debt repayments do not exceed a fraction of the future resale value of capital. Moreover, we explicitly introduce the used capital market into the model. Producers can choose between used capital and new capital. New capital is produced with consumption goods. After production, a fraction of new capital becomes used capital and then traded on the secondary market. Relative to new capital, used capital has a higher depreciation rate, and thus higher user cost. However, it is cheaper upfront since it requires a lower down payment and thus is easier to finance.

In our model, financial constraints can distort producers' decisions through two channels: on the extensive margin, they distort firms' entry into the high-productivity modern sector and thus reduce the productivity of individual producers; on the intensive margin, they force firms within the modern sector to invest less than their optimal level, thus generate capital misallocation across firms. Introducing the market for used capital can mitigate the distortions on both margins. As shown in Rampini (2019) and Lanteri and Rampini (2023), although used capital has higher user costs (due to higher depreciation or maintenance costs) compared with new capital, it requires a lower down payment and thus is cheaper upfront and easier to finance. This feature makes it particularly attractive to financially constrained firms. With the used capital market, firms with lower net worth levels can enter the modern sector by purchasing cheaper used capital. Constrained firms in the modern sector can also invest more by utilizing used capital. Collectively, those firm-level mitigation effects lead to a significant aggregate productivity gain on the macro-level.

We start our analysis by studying an analytical model to demonstrate the mechanism. In the stationary equilibrium of the model, we analytically show that only producers with net worth above certain threshold will choose to enter into the modern sector. Moreover, we also show that, within the modern sector, highly constrained (low net worth) producers only invest in used capital; mildly constrained producers invest in both used capital and new capital simultaneously; and less constrained and unconstrained firms use new capital only. To show the role of the used capital market in the mitigation of producer-level distortions, we artificially shut down the used capital market and compare it to our economy with the used capital market. We then analytically show that: on the extensive margin, used capital facilitates producers' entry into the high-productivity modern sector, such that the measure of total producers in the modern sector is larger in the economy with the used capital market; on the intensive margin, used capital market allows firms within the modern sector to invest more, thus reducing the MPK dispersion caused by misallocation. Next, with closed-form solutions for the aggregate productivity of economies with and without used capital market, we conduct an efficiency analysis by comparing aggregate productivity in these two economies. Through the comparison, we analytically prove that the model with used capital market has higher aggregate productivity (efficiency). We further decompose this efficiency gain into two components coming from the extensive and intensive margin and show both components are positive as well. It is worth noting that above results hold for any assumed distribution of initial net worth.

Next, we extend our model to a dynamic setting, in which we introduce the following additional elements: 1) persistent idiosyncratic productivity shocks of producers; 2) heterogeneous households with idiosyncratic labor income risk; 3) risk-averse preferences for both household and producers; 4) a constant growth rate of the measure of total producers and labor to quantify the mitigation effects of the used capital on TFP losses. We calibrate the model at an annual frequency to match key empirical moments in the U.S. related to the aggregate economic and firm dynamic. Our quantitative model can tightly match the aggregate moments, such as debt-to-out ratio and consumption-to-investment ratio, as well as cross-sectional moments (e.g., used capital ratio, output volatilizes). Our quantitative analysis demonstrates that about 9.3% productivity gain can be achieved by considering the market for used capital. The decomposition analysis shows that the extensive entry channel accounts for 80% of the gain. We also perform several additional sensitivity analyses and we can find our main results are robust to alternative parametrization.

Related literature Our paper is related to several stands of literature. First, it is related to the literature on the secondary market for used capital and capital reallocation.[3] Eisfeldt and Rampini (2007), Rampini (2019), Lanteri and Rampini (2023) characterize firms' optimal choices between used capital and new capital under financial frictions,

[3]To focus on the role of used capital market in mitigating firm-level distortions, we abstract from various frictions in the secondary market for real assets studied in the literature, such as search frictions (e.g., Ramey and Shapiro (2001), Gavazza (2011a,b, 2016), Ottonello (2021), Wright, Xiao, and Zhu (2020)), adverse selection (e.g., Akerlof (1978), Kurlat (2013)), and so on.

and show that financially constrained firms use more used capital. Although our model builds upon those papers, we departure from them in the following two aspects: first, we also show that used capital can facilitate entry into the high-productivity modern sector, a channel that is quantitatively more important and is supported by empirical evidence in Ma et al. (2022); second, we go one step further to analysis the macro-level impacts, that is the aggregate productivity gain generated by the used capital market. Starting with Eisfeldt and Rampini (2006), a series of papers study the cyclical pattern of capital reallocation and its macroeconomic implications (e.g., Eisfeldt and Rampini (2008), Lanteri (2018)). Our paper complements this literature by studying the implication of used capital market on aggregate productivity. To the best of our knowledge, our paper is among the first to study the role of used capital market in generating aggregate productivity gain through mitigating firm-level distortions caused by financial frictions on both extensive and intensive margin.

Our paper is also related to the literature on financial frictions and aggregate productivity (e.g, Buera, Kaboski, and Shin (2011), Midrigan and Xu (2014), Moll (2014), Buera and Moll (2015), Li and Xu (2023)).[4] The closest paper to ours in this field is Li and Xu (2023), which shows that leased capital can generate 5% aggregate productivity gain through facilitating firms' entry into the high productivity sector. Different from this paper, we focus on the role of the market for used capital, which is also important in real economy.

The rest of our paper is organized as follows. Section 2 presents our main theoretical results using an analytical model. In Section 3, we describe the setting of our quantitative dynamic model, characterize the competitive equilibrium, and present our quantitative results. Section 4 concludes.

2 An analytical model

In this section, we describe a simple analytical model with the market for used capital, collateral constraint, and sectoral choices of entrepreneurs building on Rampini (2019) and Midrigan and Xu (2014). We first analytically characterize the allocation of new and

[4]Following the seminar papers by Hsieh and Klenow (2009) and Restuccia and Rogerson (2008), a series of papers examine the role of financial frictions in generating capital miallocation with in sector, including Gopinath, Kalemli-Özcan, Karabarbounis, and Villegas-Sanchez (2017), David and Venkateswaran (2019), Cavalcanti et al. (2021), Kehrig and Vincent (2021), Hu, Li, and Xu (2020) among others. More recent papers, including Asker, Collard-Wexler, and De Loecker (2014), David, Hopenhayn, and Venkateswaran (2016), Haltiwanger et al. (2018), Whited and Zhao (2021), David, Schmid, and Zeke (2022), Edmond, Midrigan, and Xu (2023) further study other sources of capital misallocation.

used capital and sectoral choices of firms with different levels of net worth in the presence of financial frictions. Then, by comparing our benchmark case with the market for used capital and a case without it, we show that the ability for producers to purchase used capital facilitates more entry into the productive sector and mitigates MPK dispersion within the productive sector, thus generating efficiency gains in the economy.

2.1 Model environment

This is a discrete time setting. The model economy consists of a representative household and heterogeneous producers. The representative household is infinitely lived, while producers only live for two periods, young and old.[5] They all maximize their lifetime utility derived from consumption. For simplicity, we assume that both households and producers are risk-neutral. This assumption is relaxed in our quantitative analysis in Section 3.

In each period, a continuum of old producers die, and the same amount of young producers is born. As in Midrigan and Xu (2014), producers can operate either in a traditional sector or in a modern sector. The traditional sector uses only labor and an unproductive technology to produce, while the modern sector produces with both labor and capital using a more productive technology. When producers are young, they choose one of the two sectors to enter. Entry into the modern sector requires an up-front fixed cost. Producers also need to make the borrowing or saving decisions. They can borrow from the representative household or other saving producers at a risk-free rate R, but borrowing is subject to a collateral constraint. Once a producer decides to enter the modern sector, she can use her own net worth and borrowing to purchase either new capital or used capital. This capital is then used for production in the next period. After production, undepreciated new capital can be sold as used capital at the market price in the secondary market, while used capital fully depreciates.[6] In the end, the old producers consume the output from production and resale value of capital after paying back the debt.

[5]This two-period overlapping generation setting allows us to essentially solve a simple two-period problem in stationary equilibrium, as in Rampini (2019). Moreover, in this setting, the resale value of new capital, which is also the price of old capital is positive, given that a new generation of producers will have demand for used capital. However, if we directly set up a two-period model, the resale value of new capital in the second period will be zero, since it is the end of the world. As a result, the market for used capital can not exist.

[6]This setting follows Rampini (2019) and Lanteri and Rampini (2023). It captures the fact the new capital is more durable than used capital.

2.1.1 Household

The infinitely-lived and risk-neutral representative household maximizes its lifetime utility from consumption. The utility function is given by

$$\sum_{t=0}^{\infty} \beta^t C_t^h,$$

where $\beta \in (0, 1)$ is the discounted factor and C_t is consumption. The household also saves B_t^h in each period. Accordingly, the household's budget constraint at time t is given by:

$$C_t^h + B_t^h = R_{t-1} B_{t-1}^h,$$

where R_{t-1} is the gross interest rate on saving from time $t-1$ to t. Since this economy has no aggregate risk, the household's optimal saving decision implies that $R_t = \frac{1}{\beta}$ for all t.

2.1.2 Producers

In each period, the economy is populated by a measure of old producers and new-born young producers. Old producers produce output based on the sectoral and investment choices that they made when they were young, repay the debt, sell the used capital in the secondary market, and consume the remaining goods. Young producers are born with exogenous net worth w distributed over the interval $[w_{min}, w_{max}]$ according to an exogenous non-degenerate distribution $\pi(w)$. We index each producer with i and denote their age (i.e., young or old) by subscript 0 and 1, respectively. We omit the producer index i wherever appropriate for simplicity. In each period, young producers with heterogeneous net worth make a series of optimal decisions, including the choice between traditional and modern sectors, saving and borrowing decisions, and investment decisions conditional on entry into the modern sector.

Traditional Sector: Young producers that choose the traditional sector in period $t-1$ face a technology that produces output Y_t^{tra} without capital in period t:

$$Y_t^{tra} = z_t^{1-\alpha}, \tag{1}$$

where $\alpha \in (0, 1)$ is the degree of returns to scale, z_t is the idiosyncratic productivity in period t, and superscript tra denotes the traditional sector. The production function (1) can be interpreted in the way that producers produce with inelastic labor, which is normalized to 1 for simplicity.

The problem of young producers in this sector is to maximize their lifetime utility given by

$$\max C_{0,t}^{tra} + \beta E_t \left(C_{1,t+1}^{tra} \right), \tag{2}$$

subject to:

$$C_{0,t}^{tra} = B_t^{tra} + w_{0,t}, \tag{3}$$

$$C_{1,t+1}^{tra} = Y_{t+1}^{tra} - R_t B_t^{tra}, \tag{4}$$

$$B_t^{tra} \leq 0, \tag{5}$$

$$C_{0,t}^{tra}, C_{1,t+1}^{tra} \geq 0, \tag{6}$$

where $C_{0,t}^{tra}$ and $C_{1,t+1}^{tra}$ denote producers' consumption when they are young at time t and when they become old at time $t+1$. B_t^{tra} is the debt position of the producer. These producers are unable to borrow, so $B_t^{tra} \leq 0$ implies that they are saving money, and thus are financially unconstrained.

Modern Sector: Producers in the modern sector (denote by *mod*) have access to a more advanced production technology. It differs from that of the traditional sector in two ways: first, it requires capital as input; second, it also uses inputs more efficiently. There are two types of capital that producers can use, i.e., new capital and used capital. New capital can be produced by one unit of consumption goods, thus having a price equal to 1. After production, new capital becomes used capital and then is traded in the secondary market at price q_t at time t. The production function is given by

$$Y_t^{mod} = (\kappa z_t)^{1-\alpha} \left(K_{t-1}^o + K_{t-1}^n \right)^{\alpha}, \tag{7}$$

where $\kappa > 1$ is the relative productivity gap between the modern sector and the traditional sector, K_{t-1}^o and K_{t-1}^n are the amount of used capital and new capital, which are determined at the end of $t-1$. $K_{t-1} = K_{t-1}^o + K_{t-1}^n$ is the total amount of capital used at time t. Following Eisfeldt and Rampini (2007), Rampini (2019) and Lanteri and Rampini (2023), used capital and new capital are assumed to be perfect substitutes in production.

At time t, if a young producer chooses to enter the modern sector, he must pay a fixed entry cost f. The producer uses his own net worth $w_{0,t}$ together with borrowed money B_{t+1}^{mod} to cover the entry cost, to purchase used capital or new capital for production in the next period, and to consume. The amount of money the produce can borrow is tightly linked to the amount of capital he purchases through a collateral constraint. As in Rampini (2019), the collateral constraint requires that debt repayments do not exceed a fraction

$\theta \in (0,1]$ of the future resale value of capital. Given that we assume used capital fully depreciated after production, thus it has no resale value. Therefore, producers can only borrow against the future resale value of new capital, which does not depreciate and can be sold as used capital in the secondary market. That said, the collateral constraint is given by [7]

$$R_t B_t^{mod} \leq q_{t+1} \theta K_t^n, \tag{8}$$

where q_{t+1} is the price of used capital at time $t+1$, θ captures the tightness of the financial constraint.

Similar to producers in the traditional sector, producers who choose to enter the modern sector also maximize their lifetime utility derived from consumption. Their maximization problem is given by

$$\max C_{0,t}^{mod} + \beta E_t \left(C_{1,t+1}^{mod} \right), \tag{9}$$

subject to:

$$C_{0,t}^{mod} = B_t^{mod} + w_{0,t} - K_t^n - q_t K_t^o - f, \tag{10}$$

$$C_{1,t+1}^{mod} = Y_{t+1}^{mod} - R_t B_t^{mod} + q_{t+1} K_t^n, \tag{11}$$

$$K_t^n, K_t^o, C_{0,t}^{mod}, C_{1,t+1}^{mod} \geq 0, \tag{12}$$

and collateral constraint in equation (8).

2.1.3 Market clearing conditions

In a competitive equilibrium, markets for used capital, consumption goods, and debt are all clear. Used capital clearing implies that

$$\int K_t^o di = \int K_{t-1}^n di. \tag{13}$$

Consumption goods market clearing implies that

$$\int Y_t^{tra} di + \int Y_t^{mod} dj = C_t + \int \left(C_{0,t}^{tra} + C_{1,t}^{tra} \right) di + \int \left(C_{0,t}^{mod} + C_{1,t}^{mod} \right) di + \int K_t^n di + \int f di. \tag{14}$$

[7] Rampini and Viswanathan (2010, 2013) show how to derive such collateral constraints in an economy with limited enforcement without exclusion, in which firms can default on their promises and retain their out, a fraction $1 - \theta$ of their capital, and access to the markets for capital goods and financing.

Debt market clearing implies that

$$B_t^h + \int B_t^{tra} di = \int B_t^{mod} di. \tag{15}$$

2.2 Equilibrium characterization

In this section, we first define the stationary competitive equilibrium. We then discuss the optimal choice between used capital and new capital of producers that already choose to enter the modern sector and the optimal decision of entering the modern sector. Finally, we use a numerical example to illustrate the properties of stationary competitive equilibrium.

2.2.1 Stationary competitive equilibrium

To analytically characterize the optimal decisions, we make three simplifying assumptions. First, we assume that all producers have the same idiosyncratic productivity z_t. The only heterogeneity across them is their initial net worth $w_{0,t}$ (we use w for simplicity in the following analysis) drawn from an exogenous non-degenerate distribution $\pi(w)$. Second, we assume that young producers can observe their next period's idiosyncratic productivity z in advance, and then make decisions on investment, borrowing, and sectoral choices before z is realized.[8] As a result of this assumption, we remove the conditional expectation operator in producers' maximization problems. Third, to derive analytical results of the sector choice decisions, we assume that producers only consume when they are old, i.e., $C_{0,t}^{tra} = C_{0,t}^{mod} = 0$. Based on these simplifying assumptions, we then set up the Lagrangian function of producers in both the traditional sector and modern sector, and derive the optimal conditions in Appendix A.1.

A *stationary competitive equilibrium* is a set of policy functions mapping initial net worth w to an allocation $\{C_0^{tra}(w), C_1^{tra}(w), C_0^{mod}(w), C_1^{mod}(w), K^n(w), K^o(w), B^h, B^{tra}(w), B^{mod}(w)\}$, that is, consumption of entrepreneurs in traditional sector when they are young and old, consumption of producers in modern sector when they are young and old, new capital investment, old capital investment, and debt choices of household and producers in traditional sector and modern sector, and a price of old capital q, such that household and producers maximize their lifetime utility from consumption, and the markets for

[8] This assumption of "observing idiosyncratic productivity in advance "is standard in the investment literature, see Moll (2014) and Midrigan and Xu (2014). It can be justified by the fact that producers have insider information.

consumption goods, used capital, and debt all clear. Since we are considering the stationary competitive equilibrium, we omit the time subscripts of all variables in the following analysis of Section 2.

2.2.2 Optimal choice between used capital and new capital in the modern sector

In this subsection, we characterize the optimal choice between used capital and new capital of producers that have already decided to enter the modern sector. Following Rampini (2019), to characterize the optimal choice between new and used capital, it is useful to define two terms for both new capital and used capital, the user cost of capital and the down payment. We start with unconstrained firms and then discuss the case of constrained firms.

A firm is defined as a constrained firm when its borrowing constraint is binding. For an unconstrained firm, the frictionless user cost in the language of Jorgenson (1963) for new capital is equal to $u_n = 1 - \beta q$, the current price 1 minus the discounted resale value βq. While the user cost of used capital is $u_o = q$ because used capital has no resale value by assumption. To assess the financing needs of a unit of capital, we further define the down payment of each type of capital, which measures the minimal amount of net worth that the firm needs to purchase a unit of capital. For one unit of new capital, the firm can borrow $\beta\theta q$ fraction against it, so the down payment per unit of new capital is given by $\psi_n = 1 - \beta\theta q$, the price of the asset minus the present value of the fraction of the resale value of the depreciated capital that the firm can pledge. Firms can not borrow when they purchase used capital since there is no resale value in the next period, the down payment of used capital is $\psi_o = q$. Based on the above definitions, we can see that for new capital, the down payment ψ_n is larger than its user cost u_n for an unconstrained firm, given $0 < \theta < 1$, that is $\psi_n > u_n$. For used capital, the down payment is equal to the user cost, as well as the price of capital, that is $\psi_o = u_o = q$.

Next, we discuss the case in which the firm can be financially constrained. Our goal is to show that in the presence of financial frictions, both user cost and down payment play a role in determining the optimal choice of capital. To do so, we plug the user cost and down payment of two types of capital into their corresponding optimal conditions. In the stationary equilibrium, according to the derivation in Appendix A.1.1, the first-order conditions (FOC) for new and used capital are as follows,

$$\beta\mu_1 \left(\frac{\partial Y^{mod}}{\partial K^n} + q \right) + \beta\lambda\theta q + v^n = \mu_0, \tag{16}$$

$$\beta\mu_1\left(\frac{\partial Y^{mod}}{\partial K^o}\right)+v^o=\mu_0 q, \tag{17}$$

where μ_0 and $\beta\mu_1$ are the Lagrangian multipliers of budget constraints, $\beta\lambda$ is the multiplier of the collateral constraint, ν^n and ν^o are the multipliers of the non-negativity constraint for new and old capital, respectively.

Using the down payment for an unconstrained firm, we can rewrite the FOCs of two types of capital as following investment Euler equations that pin down the optimal choice of capital:

$$1=\beta\frac{\mu_1}{\mu_0}\frac{\frac{\partial Y^{mod}}{\partial K^n}+(1-\theta)q}{\psi_n}+\frac{\nu_n}{\mu_0\psi_n}, \tag{18}$$

$$1=\beta\frac{\mu_1}{\mu_0}\frac{\frac{\partial Y^{mod}}{\partial K^n}}{\psi_o}+\frac{\nu_o}{\mu_0\psi_o}. \tag{19}$$

Moreover, using the optimal condition $\mu_0 = \mu_1+\lambda$ of debt B^{mod} (see derivation in Appendix A.1.1) to substitute out μ_0 in above two Euler equations, we have following relationship: 1) for the Euler equation of new capital in equation (18), dividing both side by μ_1 and rearranging terms, we can obtain

$$\underbrace{1-\beta q}_{u_n}+\frac{\lambda}{\mu_1}\underbrace{(1-\beta\theta q)}_{\psi_n}\geq\beta\frac{\partial Y^{mod}}{\partial K^n},$$

where inequality holds because $v^n \geq 0$; 2) for the Euler equation of used capital in equation (19), dividing both sides by μ_1 and rearranging terms, we can obtain

$$\underbrace{q}_{u_o}+\frac{\lambda}{\mu_1}\underbrace{q}_{\psi_o}\geq\beta\frac{\partial Y^{mod}}{\partial K^o},$$

where the inequality holds because $v^o \geq 0$. Taken together, we have a following general condition for both new and used capital,

$$u_j+\frac{\lambda}{\mu_1}\psi_j\geq\beta\frac{\partial Y^{mod}}{\partial K^j}, \tag{20}$$

which holds for at least one type of capital with equality whether or not the firm is constrained, since total $K>0$ and hence $K^j>0$ for at least one j in n,o. It implies that the discounted marginal product of capital $\beta\frac{\partial Y^{mod}}{\partial K^j}$ equals the frictionless user costs μ_j plus a penalty for the down payment ψ_j when the borrowing constraint binds. As a result, in the presence of financial frictions, the optimal capital choice is determined by the frictionless user costs and the down payment when the firm is financially constrained. Based on equation (20), we can derive a set of equilibrium conditions that can be summarized by the following proposition.

Proposition 1. *In a stationary equilibrium in which both types of capital are used, we must have:*

1. *The user cost of new capital for an unconstrained firm has to be less than or equal to the user cost of used capital, that is $u_n \leq u_o$;*
2. *The down payment on new capital has to strictly exceed the down payment on used capital, that is $\psi_n > \psi_o$;*
3. *The price of used capital satisfies $\frac{1}{1+\beta} \leq q < \frac{1}{1+\beta\theta}$ in equilibrium.*

Proof. See Appendix A.2. □

Based on the results in Proposition 1, we now discuss how the user cost and down payment difference across two types of capital affect firms' optimal choice between new capital and used capital. We start by defining the user cost of capital for a potentially constrained firm. From equation (18) and (19), we define the user cost of new capital and used capital for a potentially constrained firm with net worth w_0 as

$$u_n(w_0) = u_n + \beta \frac{\lambda}{\mu_0} (1-\theta) q = \beta \frac{\mu_1}{\mu_0} \frac{\partial Y^{mod}}{\partial K^n} + \frac{\nu_n}{\mu_0}, \tag{21}$$

$$u_o(w) = u_o = \beta \frac{\mu_1}{\mu_0} \frac{\partial Y^{mod}}{\partial K^o} + \frac{\nu_o}{\mu_0}. \tag{22}$$

Next, we consider two extreme cases to show how firms with different financial constraint levels make their optimal investment choices. First, let us consider an unconstrained firm with a high net worth w, whose multiplier on the borrowing constrain $\lambda = 0$. As a result, its user cost is just the frictionless user cost $u_n(w) = u_n$. Given we have shown that $u_n \leq u_o$ in Proposition 1, unconstrained firms never invest in used capital. In this case, unconstrained firms evaluate two types of capital based on their frictionless user cost. Used capital is dominated by new capital from the perspective of unconstrained firms. Second, let us consider a severely constrained firm with a very low net worth (w close to 0), whose multiplier on the borrowing constrain $\lambda \to \infty$. To characterize this firm's between new can used capital, we rewrite equation (21) as

$$u_n(w) = \psi_n - \beta \frac{\mu_1}{\mu_1 + \lambda} (1-\theta) q, \tag{23}$$

As $\lambda \to \infty$, $\beta\frac{\mu_1}{\mu_1+\lambda} \to 0$, thus $u_n(w) \to \psi_n$, meaning that the user cost is the down payment as the firm becomes severely constrained. As a result, severely constrained firms evaluate the types of capital simply based on the required down payments, because they discount the part of the residual value they recover the next period completely. Since used capital requires a lower down payment than new capital $\psi_o < \psi_n$ by Proposition 1, such firms choose to adopt used capital because used capital involves smaller financing needs in terms of internal funds.

Based on the analysis of the above two extreme cases, we summarize the optimal investment choices between new and used capital for firms with different levels of net worth in Proposition 2.

Proposition 2. *In a stationary equilibrium in which both types of capital are used, the optimal choices between new and used capital for producers in the modern sector are characterized as follows:*

1. *If $q > \frac{1}{1+\beta}$, there exists threshold $\underline{w}_n < \bar{w}^o < \bar{w}$ such that: firms with $w \leq \underline{w}_n$ invest only in used capital; firms with $w \in (\underline{w}_n, \bar{w}^o)$ invest in both new and old capital; firms with $w \in [\bar{w}^o, \bar{w})$ invest only in new capital but are still financially constrained; firms with $w \geq \bar{w}$ invest only in new capital and are unconstrained.*

2. *If $q = \frac{1}{1+\beta}$, then $\bar{w}^o = \bar{w}$, such that firms with $w \leq \underline{w}_n$ invest only in used capital; firms with $w \in (\underline{w}_n, \bar{w}^o)$ invest in both new and used capital; firms with $w \geq \bar{w} = \bar{w}^o$ invest only in new capital and achieve optimal scale of production.*

Proof. See Appendix A.3. □

Proposition 2 demonstrates how producers' initial net worth determines their tightness of financial constraint, in turn, affects their choices between used capital and new capital. In equilibrium, the price of used capital q is determined by the market clearing condition in equation (13). Its actual value depends on the distribution of net worth across firms.

With the results in Proposition 2, we can characterize the marginal product of capital (MPK) as well as the total amount of capital by producers in the modern sector with different levels of net worth. Given the production function in equation (7), the MPK can be written as

$$MPK = \alpha\left(\kappa z\right)^{1-\alpha} K^{\alpha-1}, \tag{24}$$

where $K = K^o + K^n$ is the total utilized capital. Using the results in Proposition 2, we can derive the MPK and total utilized capital for producers with different levels of net worth below:

Lemma 1. *Each producer's total utilized capital can be calculated as follows:*

$$K = \begin{cases} \frac{w-f}{q} & w \leq \underline{w}_n \\ \left(\frac{q\left(1+\frac{1-\beta q-q}{q(1+\beta\theta)-1}\right)}{\beta\alpha(\kappa z)^{1-\alpha}} \right)^{\frac{1}{\alpha-1}} & w \in (\underline{w}_n, \bar{w}^o) \\ \frac{w-f}{1-\beta\theta q} & w \in [\bar{w}^o, \bar{w}) \\ \left(\frac{1-\beta q}{\beta\alpha(\kappa z)^{1-\alpha}} \right)^{\frac{1}{\alpha-1}} & w \geq \bar{w} \end{cases}.$$

Moreover, each producer's used capital and new capital can be calculated as follows:

$$K^o = \begin{cases} \frac{w-f}{q} & w \leq \underline{w}_n \\ \frac{1-\beta q\theta}{1-\beta q\theta-q} \left(\frac{q+\frac{1-\beta q-q}{q(1+\beta\theta)-1}q}{\beta\alpha(\kappa z)^{1-\alpha}} \right)^{\frac{1}{\alpha-1}} - \frac{w-f}{1-\beta q\theta-q} & w \in (\underline{w}_n, \bar{w}^o) \\ 0 & w \in [\bar{w}^o, \bar{w}) \\ 0 & w \geq \bar{w} \end{cases}.$$

$$K^n = \begin{cases} 0 & w \leq \underline{w}_n \\ \frac{-q}{1-\beta q\theta-q} \left(\frac{q+\frac{1-\beta q-q}{q(1+\beta\theta)-1}q}{\beta\alpha(\kappa z)^{1-\alpha}} \right)^{\frac{1}{\alpha-1}} + \frac{w-f}{1-\beta q\theta-q} & w \in (\underline{w}_n, \bar{w}^o) \\ \frac{w-f}{1-\beta\theta q} & w \in [\bar{w}^o, \bar{w}) \\ \left(\frac{1-\beta q}{\beta\alpha(\kappa z)^{1-\alpha}} \right)^{\frac{1}{\alpha-1}} & w \geq \bar{w} \end{cases}.$$

The MPK of producers in the modern sector can be calculated as follows:

$$MPK = \begin{cases} \alpha\left(\kappa z\right)^{1-\alpha} \left(\frac{w-f}{q} \right)^{\alpha-1} & w \leq \underline{w}_n \\ q\left(1+\frac{1-\beta q-q}{q(1+\beta\theta)-1}\right)\frac{1}{\beta} & w \in (\underline{w}_n, \bar{w}^o) \\ \alpha\left(\kappa z\right)^{1-\alpha} \left(\frac{w-f}{1-\beta\theta q} \right)^{\alpha-1} & w \in [\bar{w}^o, \bar{w}) \\ (1-\beta q)\frac{1}{\beta} & w \geq \bar{w} \end{cases}.$$

According to Lemma 1, when the producer is sufficiently constrained with net worth

$w \leq \underline{w}_n$, it only uses used capital, and the amount of used capital is $\frac{w-f}{q}$. As a result, its MPK varies (is decreasing) in net worth w. When the producer is constrained but with a higher net worth $w \in (\underline{w}_n, \bar{w}^o)$, it starts to use both used capital and new capital, the MPK is independent of the net worth level. When the producer's net worth $w \in [\bar{w}^o, \bar{w})$ but is still constrained, it invests only in new capital with the amount given by $\frac{w-f}{1-\beta\theta q}$. In this case, its MPK decreases with the amount of new capital (thus the amount of net worth). Finally, when the produce's net worth $w \geq \bar{w}$ is unconstrained, it achieves the optimal investment scale by equalizing its MPK to the user cost of new capital. In Section 2.2.4, we consider a numerical example and use Figure 1 to illustrate how a producer's capital investment and MPK change with its net worth.

[**Place Figure 1 about here**]

2.2.3 Choice between the traditional sector and modern sector

Next, we study the choice between the traditional sector and the modern sector. To do so, we first derive a producer's potential consumption in the modern sector and traditional sector. Then, we compare the utility level that the producer can get from these two sectors. A utility maximization producer will choose the sector that generates the highest utility.

For those newly born producers, if it chooses the traditional sector, its consumption is given by equation (3) and (4). Given our assumption that $C_0^{tra} = 0$, the producer's consumption when it is old is thus given by

$$C_1^{tra} = z^{1-\alpha} + \beta^{-1} w. \tag{25}$$

As a result, the producer's total utility is βC_1^{tra}.

However, if the producer chooses to enter the modern sector, its consumption is given by equation (10) and (11). Assuming $C_0^{mod} = 0$, then the producer's consumption when it is old is given by

$$C_1^{mod} = (\kappa z)^{1-\alpha} (K^o + K^n)^\alpha - \beta^{-1} q K^o - (\beta^{-1} - q) K^n - \beta^{-1} f + \beta^{-1} w, \tag{26}$$

where the choice of capital (K^o and K^n) depends on producers' initial net worth as discussed in Lemma 1. In this case, the producer's total utility is thus βC_1^{mod}.

Next, we characterize the producer's choice between the modern sector or the traditional sector by comparing the total utility it can get from those two sectors, i.e.,

βC_1^{tra} and βC_1^{mod}. The producer will choose to enter the modern sector if it gets higher utility, that is $\beta C_1^{mod} > \beta C_1^{tra}$. Given the linear utility function, the comparison of utility is equivalent to the comparison of consumption level. We define $\triangle = C_1^{mod} - C_1^{tra}$, which is the value difference between choosing the traditional sector and choosing the modern sector. It is immediate that $\triangle$ is a function of producers' net worth w. To analyze the $\triangle$ in detail, we first need to calculate C_1^{mod} for producers with different levels of net worth by plugging the optimal investment results in Lemma 1 into equation (11).

There are a total of 4 cases to consider if the producer chooses the modern sector: 1) if $w \leq \underline{w}_n$, the producer is sufficiently constrained and only invests in used capital, we denote its time 1 consumption by $C_1^{mod,o}$; 2) if $w \in (\underline{w}_n, \bar{w}^o)$, the producer is constrained but invests in both used capital and new capital, we denote its time 1 consumption by $C_1^{mod,on}$; 3) if $w \in [\bar{w}^o, \bar{w})$, the producer is constrained and invests only in new capital, we denote its time 1 consumption by $C_1^{mod,nc}$; 4) if $w \geq \bar{w}$, the producer is unconstrained and only invests in new capital, we denote its time 1 consumption by $C_1^{mod,nu}$. The detailed expressions are calculated in Appendix A.4.1. As a result, for a producer with initial net worth w, $\triangle$ function will consist of 4 parts depending on the value of initial net worth w:

$$\triangle = \begin{cases} \triangle_o = C_1^{mod,o} - C_1^{tra} & w \leq \underline{w}_n \\ \triangle_{on} = C_1^{mod,on} - C_1^{tra} & w \in (\underline{w}_n, \bar{w}^o) \\ \triangle_{nc} = C_1^{mod,nc} - C_1^{tra} & w \in [\bar{w}^o, \bar{w}) \\ \triangle_{nu} = C_1^{mod,nu} - C_1^{tra} & w \geq \bar{w} \end{cases}.$$

The explicit expressions of $\triangle$ for each case are derived in Appendix A.4.2. Before we analyze the $\triangle$ function in detail and characterize the sector decisions, we discuss the reasonable parameter regions in the following assumption. We make this assumption on the fixed cost parameter f to ensure that there are producers in both the traditional sector and the modern sector in equilibrium.

Assumption 1. *Given the productivity z and the price of used capital q, the entry fixed cost f must satisfy: $f \in (w_{min}, f_{max})$, so that firms with low net worth w_{min} will choose the traditional sector, whereas firms with large net worth will choose to enter the modern sector. f_{max} is defined by Equation (A.25) in Appendix A.5.*

When the parameter value satisfies Assumption 1, we plot the $\triangle$ function against the initial net worth level w in Figure 1. The details for the shape of $\triangle$ function are shown in Appendix A.4.3. It is clear that in the case with the market for used capital: 1) for $w \leq \underline{w}_n$ and $w \in [\bar{w}^o, \bar{w})$, $\triangle$ is increasing and concave in net worth w; 2) for $w \in (\underline{w}_n, \bar{w}^o)$, the $\triangle$

is linearly increasing in w; 3) for $w \geq \bar{w}$, producers become unconstrained, the $\triangle$ reaches the highest value and becomes flat. Furthermore, we can see that $\triangle_{on}$ is a tangent line to both $\triangle_o$ and $\triangle_{nc}$, and $\underline{w}_n$ and $\bar{w}^o$ are the x-coordinate of the tangent points.

Based on the analysis of the $\triangle$ function, we first characterize the choice between the traditional sector and modern sector in the benchmark economy with the used capital market. Then, we artificially shut down the used capital market, analyze the optimal sector choices, and then compare the results with the benchmark economy. We summarize the results in Proposition 3.

Proposition 3. *When parameters satisfy the condition in the Assumption 1, we can obtain the following results:*

- *With the used capital market, we have:*
  - *Producers with $w < \bar{w}_m$ choose the traditional sector, where $\bar{w}_m$ is defined in equation (A.26).*
  - *Producers with $w \in [\bar{w}_m, w_{max}]$ choose the modern sector.*
  - *Whether producers in the modern sector are financially constrained and whether they invest used capital follow Proposition 2.*
- *Without the used capital market, we have:*
  - *Producers with $w < w_m$ choose the traditional sector, in which w_m in defined in equation (A.27).*
  - *Producers with $w \in [w_m, \bar{w}]$ choose the modern sector, and are still financially constrained.*
  - *Producers with $w > \bar{w}$ choose the modern sector and are unconstrained.*
- *We can prove that $f \leq \bar{w}_m < w_m < \underline{w}_n < \bar{w}^o < \bar{w}$.*

Proof. See Appendix A.6. □

Proposition 3 shows that producers' choices between the traditional sector and modern sector can be characterized analytically, by comparing their potential utility (also consumption) levels in the two sectors. In the model with a used capital market, producers with $w < \bar{w}_m$ choose the traditional sector, while producers with $w \geq w_m$ choose to enter the modern sector. Once they enter the modern sector, their investment choices are summarized in Proposition 2. In the model without a used capital market, producers can only use new capital that is harder to finance. As a result, this leads to larger distortions of producers' sectoral choices and investments. Specifically, the cutoff net worth value for producers to enter into modern sector w_m is higher in the model without used capital market than that ($\bar{w}_m$) in the model with used capital market. Intuitively, used capital, which is easier to finance, allows producers to produce with more capital and thus generate higher output, consumption, and utility. Thus, entering the modern sector becomes not only more feasible but also more attractive, such that producers with lower net worth levels are willing to enter the modern sector and invest in used capital. In other words, used capital facilitates entry into the modern sector, and the gap between $\bar{w}_m$ and w_m highlights the role of used capital in mitigating the distortion caused by financial frictions at the extensive margin. Moreover, for producers with net worth between w_m and $\bar{w}^o$, the value difference ($\triangle$ function) between the modern sector and the traditional sector is larger in the model with used capital market than that in the model without used capital market. For those producers, with a used capital market, they can have larger investments and thus generate higher output and consumption.

Overall, allowing for used capital can facilitate producers' entry into the modern sector, and allow modern producers to invest more and produce more. This benefit is more pronounced for sufficiently constrained producers.

2.2.4 A numerical example

In this section, we use a numerical example to illustrate the mechanism and the main properties of our model equilibrium. We present the key results of the numerical example in Figure 2 and describe the parameter values used for computation in the caption.

In this example, in the economy with used capital, the equilibrium price of used capital $q = 0.5082 > \frac{1}{1+\beta}$. Consistent with the results in Proposition 2 and Lemma 1, from panel (a) and (b) in Figure 2, we can see that in the economy with used capital, we have three cutoff values of net worth, that is $\underline{w}_n = 1.03, \bar{w}^o = 1.47, \bar{w} = 1.64$. Producers with net worth lower than $\underline{w}_n = 1.03$ in the modern sector only use used capital. When their new worth is between $\underline{w}_n$ and $\bar{w}^o$, they use both used capital and new capital. In this range, the total capital stock is fixed (as shown in panel (c)), but producers gradually substitute used capital with new capital as their net worth increases. Once producers' net worth becomes higher than $\bar{w}^o = 1.47$, they completely use new capital but are still financially constrained. Producers become totally unconstrained when the net worth is higher than $\bar{w} = 1.64$. Overall, in the modern sector, as the net worth increases, firms use more total capital and more new capital (panel (c)), the MPK gradually decreases (panel (d)), and the level of collateral constraint gradually decreases (panel (f)). This pattern applies to both the economy with used capital and the economy without used capital.

However, the differences between the economy with and without used capital are clear as well. As we analytically show in Proposition 3, without used capital, producers need a higher level of net worth to enter into the modern sector $w_m > \bar{w}_m$ because producers now can only use new capital that is harder to finance. This can be seen clearly in panel (e) in Figure 2, in which $w_m = 0.72$ while $\bar{w}_m = 0.52$. This entry point difference across the two economies highlights the role of used capital in facilitating more entry into the modern sector. Moreover, except for the higher level requirement for entry, by comparing the two lines in panels (c), (d), and (f) of Figure 2, we can also see that when there is no used capital market, producers in the modern sector use less total capital, have higher MPK, and are more financially constrained. These differences illustrate the role of used capital in mitigating capital misallocation within the modern sector.

To sum up, both our analytical results and the numerical example demonstrate the important role of used capital in facilitating more entry (extensive margin) into the modern sector and in mitigating capital misallocation in the modern sector (intensive margin). In the next subsection, we present our efficiency analysis in the model economy.

[**Place Figure 2 about here**]

2.3 Efficiency analysis

In this section, we analyze the efficiency gains from the used capital market by comparing the TFP in the model economy with the used capital market to a model economy without it. Through the comparison of these two economies, we analytically prove that used capital market leads to positive total efficiency gain. We further decompose this efficiency gain into two components coming from the extensive and intensive margin and show both components are positive as well.

2.3.1 The effect of used capital on TFP in the modern sector

Consistent with Midrigan and Xu (2014), our analysis focuses on producers in the modern sector since only they use capital in production. The TFP in the modern sector is calculated as:

$$TFP = \frac{\int Y_i^{mod} di}{(\int K_i di)^{\alpha}}, \tag{27}$$

where Y_i^{mod} and K_i are the output and the total utilized capital of producer i.

We further define TFP with the used capital market as $\overline{TFP}$, and the TFP without used capital market as $\widetilde{TFP}$ using equations (A.30) and (A.31), respectively. Then, the total gain from the option to use used capital can be calculated as

$$G^{TFP} = log\left(\overline{TFP}\right) - log\left(\widetilde{TFP}\right) \tag{28}$$

Based on the condition in Assumption 1, we can obtain the following proposition about the TFP gain.

Proposition 4. *When parameters satisfy the conditions in Assumption (1), compared to the economy without a used capital market, allowing for used capital market can improve TFP, i.e.,* $G^{TFP} > 0$.

Proof. See Appendix A.7. □

Proposition 4 shows that introducing the market for used capital can increase the TFP in the modern sector. Again, this is due to the effect we discussed in Proposition 3. Since used capital is easier to finance, it facilitates producers' entry into the modern sector and allows financially constrained producers to have more investment and produce more output.

2.3.2 Productivity gain decomposition

Used capital generate productivity gain via two channels: 1) by facilitating producers' entry into the modern sector; 2) by reducing losses from misallocation in the modern sector. Based on these two channels, we decompose the total TFP gain into two components accordingly:

$$G^{TFP} = G^{TFP}_{entry} + G^{TFP}_{misall}, \tag{29}$$

where G^{TFP}_{entry} is TFP gain attributed to the effects of entry, G^{TFP}_{misall} is the TFP gain from reducing capital misallocation.

To obtain G^{TFP}_{misall}, we first calculate the efficient TFP (or the first best TFP), i.e., TFP^e, for the same set of producers in the original economy.[9] To do so, we consider a hypothetical

[9] In this social planner's problem, the number of producers and the total capital stock (including both used capital and new capital) are the same as in the original decentralized economy. Social planners only reallocate the fixed amount of capital across the same set of producers to achieve the maximum output.

social planner's problem of allocating capital across these producers in order to maximize total output in the modern sector. Specifically, the planner maximizes

$$\max_{K_i} \int_{i \in m} (\kappa z)^{1-\alpha} K_i^{\alpha} di, \tag{30}$$

subject to the constraint that the total amount of capital to allocate is the same as in the original economy. The optimal allocation of capital implies that the MPK should be equalized across all producers. Here K_i is the total capital utilized by firm i, it can include both used capital and new capital. As a result, the efficient level of TFP TFP^e is given by:

$$TFP^e = \left(\int_{i \in m} (\kappa z) di \right)^{1-\alpha}, \tag{31}$$

with more details provided in Appendix A.8. Given TFP^e, it follows that the total TFP losses (in logs) from capital misallocation in the benchmark economy with the used capital market can be calculated as:

$$\overline{\Gamma_{misall}^{TFP}} = log\left(\overline{TFP^e}\right) - log\left(\overline{TFP}\right). \tag{32}$$

Similarly, when there is no used capital market, the TFP loss from capital misallocation in the benchmark economy is given by:

$$\widetilde{\Gamma_{misall}^{TFP}} = log\left(\widetilde{TFP^e}\right) - log\left(\widetilde{TFP}\right), \tag{33}$$

where $\widetilde{TFP^e}$ is the efficient level of TFP TFP^e calculated using producers in the modern sector of the economy without used capital market.

Therefore, we can calculate the TFP gain from opening the used capital market by reducing misallocation in the modern sector as:

$$G_{misall}^{TFP} = \widetilde{\Gamma_{misall}^{TFP}} - \overline{\Gamma_{misall}^{TFP}}. \tag{34}$$

With G_{misall}^{TFP} in hand, we can further calculate the TFP gain from opening the used capital market through facilitating entry as:

$$G_{entry}^{TFP} = G^{TFP} - G_{misall}^{TFP} = log\left(\overline{TFP^e}\right) - log\left(\widetilde{TFP^e}\right) \tag{35}$$

We summarize the signs of G_{misall}^{TFP} and G_{entry}^{TFP} in Proposition 5.

Proposition 5. *When parameters satisfy the Assumption 1, allowing for used capital market leads to positive gain of TFP through facilitating entry into the modern sector, that is,*

$$G_{entry}^{TFP} > 0.$$

Moreover, enabling producers to invest in used capital reduces capital misallocation among them, which in turn leads to a positive gain of TFP. So we have

$$G_{misall}^{TFP} > 0.$$

Proof. See Appendix A.8. □

The existence of a used capital market allows more producers with low levels of net worth to enter the modern sector. With decreasing return to scale production function, the increase in the number of producers in the modern sector leads to an increase in the TFP. Although this increase in the number of entrants has the tendency to raise capital misallocation because those entrants are highly constrained producers with high MPK, used capital allows producers with net worth ($w > w_m$) to use more capital, which reduces capital misallocation. The overall effect in our analytical model is still positive such that $G_{misall}^{TFP} > 0$. This is further confirmed in our quantitative model in Section 3.

3 The quantitative model

In this section, we first describe the environment of our quantitative dynamic model of investment under financial friction with the market for used capital. Then, we calibrate the model and evaluate its ability to account for key moments in the data. Finally, we conduct a quantitative analysis and present the results to demonstrate the role of the used capital market in generating aggregate efficiency gain through mitigating firm-level distortions caused by financial frictions.

3.1 Model environment

The general setting is similar to the analytical model in Section 2. Different from the analytical model, in this dynamic setting, we now assume that producers are also infinitely lived. Moreover, to generate quantitatively plausible firm dynamics and heterogeneity in producers' capital stocks that are consistent with data, we introduce the following additional elements: 1) persistent idiosyncratic productivity shocks of producers; 2) heterogeneous households with idiosyncratic labor income risk; 3) risk-averse preferences for both household and producers; 4) a constant growth rate of the measure of total producers and labor.

3.1.1 Household

The economy is populated by a measure one of households. Each household maximizes its lifetime utility:

$$E_0 \sum_{t=0}^{\infty} \beta^t log(C_t^h).$$

where β is the time discount and C_t^h is the consumption at time t. We omitted the household index here for simplicity. Households provide inelastic labor supply to producers and save asset. Their budget constraint is given by:

$$C_t^h + B_t^h = W \gamma^t v_t + R B_{t-1}^h, \tag{36}$$

where B_t^h is the household's saving, W and $R = 1 + r$ are the equilibrium wage and interest rate in this economy.[10] As in Midrigan and Xu (2014), we assume that households' effective labor grows at a constant rate γ to ensure the existence of balance growth path, and is subject to an idiosyncratic labor efficiency shock v_t. v_t reflects the uninsurable idiosyncratic labor income risk faced by households. It follows a Logarithm normal AR(1) process where $log(v_t) = \rho_w log(v_t) + \epsilon_{w,t}$, and the innovation $\epsilon_{w,t}$ follows a normal distribution.

3.1.2 Producers

The economy is also populated by a measure of M_t producers. This measure grows over time at a constant rate γ along the balanced growth path such that $M_t = \gamma^t$. Producers can operate either in the low-productivity traditional sector or the high-productivity modern sector. Entry into the modern sector requires an up-front investment in sunk entry costs. In each period, a measure $(\gamma - 1) M_t$ of new producers enter the economy. The new producers are endowed with zero net worth and operate in the traditional sector. Over

[10] As in Midrigan and Xu (2014), since there is no aggregate shock, we consider a stationary equilibrium, in which prices, e.g., wage W, interest rate R, and used capital price q, do not change overtime. As a result, in the following analysis, we omit the time subscript of them.

time, once they accumulate enough net worth, they may choose to enter the modern sector.

Traditional sector: Producers in this sector have access to a decreasing returns technology that produces output Y_t^{tra} using labor L_t as the only factor of production:

$$Y_t^{tra} = exp(z + e_t)^{1-\eta} L_t^{\eta}, \tag{37}$$

where $\eta < 1$ is the degree of returns to scale, z is the permanent component of the producer's productivity, while e_t is a transitory component that evolves according to a Markov process on $E = \{e_1, \cdots, e_T\}$ with the associated transition probability $p_{ij} = Pr(e_{t+1} = e_j | e_t = e_i)$. For those new entrants, we assume that they draw initial productivity e_t from the stationary distribution associated with p, denoted as $\bar{p}_i$. Additionally, entrants draw the permanent productivity component z from a distribution $G(z)$, whose mean we normalize to unity.

Producers in this sector maximize their lifetime utility:

$$E_0 \sum_{t=0}^{\infty} \beta^t log(C_t^{tra}),$$

where C_t^{tra} is the producer's consumption at time t. The budget constraint they face depends on whether they choose to stay in the traditional sector or switch to the modern sector. For those who choose to stay in the traditional sector, their budget constraint is given by

$$C_t^{tra} = Y_t^{tra} - WL_t^{tra} - (1 + r)B_{t-1}^{tra} + B_t^{tra}. \tag{38}$$

Since they do not have capital as collateral, they can not borrow money from the bank. Therefore, $B_t^{tra} \leq 0$, meaning that they may save money.

For those producers who choose to switch to the modern sector, their budget constraint is given by

$$C_t^{tramod} = Y_t^{tra} - WL_t^{tra} - (1 + r)B_{t-1}^{tra} + B_t^{mod} - K_t^n - qK_t^o - exp(z)f, \tag{39}$$

where $exp(z)f$ is the fixed cost for a producer to enter the modern sector, which is proportional to the permanent component z.[11] K_t^n, K_t^o represent entrepreneurs' choice of new capital and used capital at the end of period t, q is the price of used capital in

[11]This assumption ensures that even the most productive producers face a non-trivial cost of entering the modern sector. Without scaling by productivity, it would be difficult to match the size distribution of producers in the data given that only the most productive producers enter the modern sector.

equilibrium. In our setting, new capital is produced by one unit of consumption goods, while used capital comes from new capital. Specifically, in each period, after production, a fraction δ^n of new capital becomes used capital, and is then traded on the secondary market with equilibrium price q. Also, after production, a fraction δ^o of used capital depreciates.

The producers who enter the modern sector finance expenditures on its capital investment (i.e., K_t^n and qK_t^o) and entry cost $exp(z)f$ using either its internal funds or by borrowing using one-period risk-free debt. The amount the producer can borrow is limited by a collateral constraint that requires that debt repayments do not exceed a fraction of the future resale value of capital:

$$(1+r)B_t^{mod} \leq \theta E_t \left[(1-\delta^n(1-q))K_t^n + q(1-\delta^o)K_t^o \right], \tag{40}$$

where $\theta \in [0,1]$ governs the strength of financial frictions in the economy. We assume that both used capital and new capital can be pledged as collateral for borrowing.

Modern sector: Different from the traditional sector, producers in this sector can produce output with a more productive technology using both labor and capital as inputs:

$$Y_t^{mod} = exp(z + e_t + \kappa)^{1-\eta}(L_t^{1-\alpha} K_{t-1}^{\alpha})^{\eta} \tag{41}$$

where α denotes the labor share, $\kappa \geq 0$ determines the relative productivity of the modern sector. As in Rampini (2019), new capital and old capital are considered as perfect substitutes for simplicity. As a result, a producer's total capital $K_{t-1} = K_{t-1}^n + K_{t-1}^o$. The producers in this sector also maximize their lifetime utility:

$$E_0 \sum_{t=0}^{\infty} \beta^t log(C_t^{mod}),$$

where C_t^{mod} is the producer's consumption at time t. In this sector, producers choose optimal investments in new capital K_t^n and used capital K_t^o, and hire labor for production. To finance those expenditures, they can borrow B_t^{mod} at the risk-free rate, subject to the borrowing constraint in Equation (40). Their budget constraint is given by

$$C_t^{mod} + K_t^n + qK_t^o = Y_t^{mod} - WL_t^{mod} + (1-\delta^n(1-q))K_{t-1}^n + q(1-\delta^o)K_{t-1}^o - RB_{t-1}^{mod} + B_t^{mod}. \tag{42}$$

3.1.3 Stationary equilibrium

To simply the model computation, we follow Midrigan and Xu (2014) and assume that producers can observe the next period's idiosyncratic productivity e_{t+1} in advance when they make capital investment.[12] This timing assumption allows us to solve producers' maximization problem in a two-step procedure: first, we solve optimal capital and labor choices through the static profit maximization; second, taking the maximized profit as given, we solve the dynamic optimization problem by choosing the optimal level of debt and consumption.

Given this assumption, we can rewrite the producers' optimization problem of each type of producer in recursive form. We define a producer's net worth at time $t+1$ by N_t. For producers in the modern sector, their net worth $N_t = qK_t^o + K_t^n - B_t^{mod}$; for producers in the traditional sector, their net worth $N_t = -B_t^{tra}$. Since profits, output, and the optimal choice of capital and labor are homogeneous of degree one in net worth N and permanent productivity exp(z), so we re-scale all variables by $exp(z)$. We use the lowercase $x = \frac{X}{exp(z)}$ to denote the normalized variables and write the recursive problem using normalized variables. Detailed derivations are in Appendix Section B.1.

Next, we define the stationary equilibrium of this economy. Let $A = [\underline{n}, \bar{n}]$ denote the compact set of values a producer's net worth can take, and Λ denote a family of its subsets. We denote the measure of the traditional sector by $\Phi_t^{tra}(n, e)$, and the measure of the modern sector by $\Phi_t^{mod}(n, e)$. These two measures evolve according to Equations (B.10) and (B.11), as shown in Appendix Section B.2.

A *balanced growth stationary equilibrium* is a set of price systems W, q, and r. A decision rule for workers, $c^h(n, v)$, $d^h(n, v)$, for producers $c^j(n, e)$ and $n^j(n, e)$, where $j \in \{tra, tramod, mod\}$, a switching decision $\zeta(n, e)$ for producers in the traditional sector, measures of producers in traditional sector $\Phi_{t+1}^{tra}(n, e)$, in modern sector $\Phi_{t+1}^{mod}(n, e)$, as well as output, labor and investment decisions by producers, $y^{tra}(e)$, $l^{tra}(e)$, $y^{mod}(n, e)$, $l^{mod}(n, e)$, $k^n(n, e)$, $k^o(n, e)$, that maximize the objective functions of household and producers in the traditional sector and modern sector, and satisfy 1) the labor market clearing condition; 2) the used capital market clearing condition; 3) the goods market clearing condition. See Appendix Section B.3 for details.

[12]Due to this assumption, both producers' optimal capital and labor choices become static. As a result, capital stocks are no longer state variables.

3.2 Calibration

In our calibration, we set the model parameters based on U.S. data, with each period representing one year. The parameter values are presented in Table 1.[13] The discount rate, $\beta = 0.94$, is chosen to match an effective real interest rate of 1.5%.[14] The growth rate, $\gamma = 1.02$, is set to match the U.S. real GDP growth rate, which averages around 2%, following Lanteri and Rampini (2023) and others. The labor share, $\alpha = 0.33$, aligns with standard values found in the literature, such as in Midrigan and Xu (2014). The span-of-control parameter, $\eta = 0.85$, is based on estimates from Basu and Fernald (1997) and Atkeson and Kehoe (2007). We set the collateralizability parameter, $\theta = 0.38$, in line with the debt capacity preservation estimated by Rampini and Viswanathan (2013). Finally, the fixed entry cost for producers entering the modern sector, $f = 0.8$, is calibrated to ensure that producers in the modern sector account for 90% of total output.

We set the depreciation rate for new capital at $\delta^n = 6\%$, following the standard value suggested by Giandrea et al. (2022). The depreciation rate for old capital, δ^o, is 10% higher than that of new capital, resulting in an effective depreciation rate of 10%, consistent with Rampini and Viswanathan (2013). Lastly, we follow Midrigan and Xu (2014) and set the relative productivity between two sectors κ to reflect an average improvement of 20%.

Following Floden and Lindé (2001) and McKay et al. (2016), we set the AR(1) coefficient of idiosyncratic wage risk at 0.9, reflecting the persistence of the estimated wage process in the U.S. Additionally, we adjust the standard deviation of idiosyncratic wage risk innovations to align with the investment-to-output ratio. The transitory productivity process follows an AR(1) form: $e_t = \rho e_{t-1} + \epsilon_t$, where $\epsilon_t \sim N(0, \sigma_e)$. The standard deviation of the permanent component is denoted by σ_z. We calibrate ρ, σ_e, and σ_z to closely match the standard deviation of real output, growth rates, and the share of investment expenditures allocated to used capital.

[Place Table 1 about here]

Next, we solve the model using method described in Appendix B.4 and then evaluate the performance of our model by comparing its moments to those observed in the data, as shown in Table 2. Overall, the model aligns closely with the data. For example, it produces a consumption-to-investment ratio of 4.74, which is very close to the observed

[13] While we calibrate the core parameters by jointly matching moments, we describe them alongside their corresponding main targets.

[14] The effective real interest rate is calculated as $\frac{R}{\gamma}$, where γ is the balanced growth rate.

value of 4.69. Similarly, the model's debt-to-output ratio is 0.8, closely approximating the actual value of 0.9.

We also analyze key moments related to standard deviations and autocorrelations. The model-generated standard deviations of labor and labor growth closely match those observed in the data. Additionally, the model effectively replicates the autocorrelations for output, total capital, and labor, with values broadly consistent with empirical findings. This demonstrates that the model captures the dynamic behavior of the economy, particularly in terms of volatility and persistence in key variables, in alignment with the data.

[**Place Table 2 about here**]

3.3 Quantitative results

In this subsection, we first outline the methods used to calculate the total factor productivity (TFP) loss due to misallocation and the relative TFP gain associated with the option to a used capital. We then explore and explain our quantitative result and argue that quantitatively our used-capital facilitating channel exists and is significant. Finally, we argue that our quantitative findings are robust across a wide range of core parameter values.

3.3.1 TFP loss calculation

We first provide the method to quantify the role of used capital in mitigating the total factor productivity (TFP) loss and then decompose the loss further into intensive and extensive margins. Following the logic from equations (27) and (31), we define TFP and the efficient TFP (TFP^e) in the modern sector as follows:

$$TFP = exp(\kappa)^{1-\eta} \frac{\left[\int e_i (MPK_i)^{\frac{\alpha\eta}{\eta-1}} di\right]^{1-(1-\alpha)\eta}}{\left\{\int e_i (MPK_i)^{\frac{(1-\alpha)\eta-1}{1-\eta}} di\right\}^{\alpha\eta}}, \tag{43}$$

$$TFP^e = \left(exp(\kappa)\int e_i di\right)^{1-\eta}. \tag{44}$$

The efficient TFP (TFP^e) represents the maximum output achievable by reallocating resources optimally, as viewed from the perspective of a social planner.

The total TFP gain from used capital, denoted as G^{TFP}, can be decomposed into two components:

$$G^{TFP} = G^{TFP}_{misall} + G^{TFP}_{entry}, \tag{45}$$

where G^{TFP}_{misall} represents the TFP gain from reducing misallocation (i.e., intensive margin), and G^{TFP}_{entry} reflects the TFP gain from facilitating new entry (i.e., extensive margin).

The details for these three components are summarized as follows:

(1). Total TFP gain from used capital:

$$G^{TFP} = log(\overline{TFP}) - log(\widetilde{TFP}), \tag{46}$$

where $\overline{TFP}$ is the TFP calculated in an economy with used capital, and $\widetilde{TFP}$ is the TFP in an economy without used capital, following equation (43).

(2). TFP gain from reducing misallocation of the used capital (i.e., intensive margin):

$$G^{TFP}_{misall} = \underbrace{\left[log\left(\widetilde{TFP^e}\right) - log\left(\widetilde{TFP}\right)\right]}_{\widetilde{\Gamma^{TFP}_{misall}}} - \underbrace{\left[log\left(\overline{TFP^e}\right) - log\left(\overline{TFP}\right)\right]}_{\overline{\Gamma^{TFP}_{misall}}}, \tag{47}$$

where $\overline{TFP^e}$ and $\widetilde{TFP^e}$ are calculated with and without used capital, respectively, following equation (44).

(3). TFP gain from facilitating the entry of the used capital (i.e., extensive margin):

$$G^{TFP}_{entry} = G^{TFP} - G^{TFP}_{misall} = log\left(\overline{TFP^e}\right) - log\left(\widetilde{TFP^e}\right). \tag{48}$$

This decomposition enables us to evaluate the relative magnitude to which used capital enhances TFP via the intensive margin and extensive margin.

3.3.2 Baseline Model

The quantitative results are presented in Table 3. The column labeled "Benchmark" represents our baseline economy, while "Yes" refers to the economy with used capital and "No" to the economy without used capital. As shown in the table, the TFP loss due to misallocation is approximately 3%, aligning with findings from previous studies, such as Midrigan and Xu (2014). These results demonstrate that our analysis effectively captures the role of used capital.

Used capital significantly reduces total factor productivity (TFP) loss by alleviating financial friction on both the intensive and extensive margins. As shown in the "Benchmark", the TFP gain achieved through the entry channel is 7.63%, far exceeding the 1.65% gain from the misallocation channel (intensive margin). This highlights that the primary benefits of used capital stem from improvements on the extensive margin, underscoring its role in facilitating entry and boosting productivity. Figure 3 illustrates that the availability of used capital lowers the required net worth for transitioning from the traditional sector to the modern sector. This access enables high-productivity but low-net-worth firms to enter the modern sector more quickly, bypassing the need to accumulate significant net worth to cover fixed costs and high capital expenditures.

With the option to use used capital, more high-productivity entrepreneurs choose to enter the modern sector. These entrepreneurs are better positioned to accumulate wealth over time and require less reliance on debt in the long run. As a result, the debt intensity in the modern sector decreases to 0.83, and the fraction of constrained firms moderates to 0.96 along the balanced growth path. This shift not only boosts production in the modern sector (from 0.75 to 0.93) but also leads to a significant increase in total output (consumption) from 1.12 (0.92) to 2.20 (1.75). This improvement is driven by the 20% higher average productivity in the modern sector compared to the traditional sector.

3.3.3 Robustness

Our counter-factual experiments for producers' collateralizability θ reveal that the mitigation effects of used capital are especially vital for financially constrained producers. Specifically, TFP improvements from reducing misallocation due to used capital are 2.37% for $\theta = 0.33$ and 1.02% for $\theta = 0.50$. More importantly, used capital significantly enhances TFP by facilitating producer entry, yielding 18.6% gain for $\theta = 0.33$ and 3.33% gain for $\theta = 0.50$.

[Place Figure 3 about here]

Furthermore, we observe that considering used capital significantly increases the fraction of producers in the modern sector, as well as consumption and output. Specifically, in the "Benchmark" scenario, the fraction of producers in the modern sector rises from 0.54 to 0.72. Consumption increases from 0.91 to 0.98, while output grows from 1.21 to 1.68 when used capital is available. These findings further emphasize the critical importance of the used capital in enhancing overall economic activity. Additionally, our counterfactual experiments for $\theta = 0.25$ and $\theta = 0.75$ reveal that as producers become more financially constrained, the utilization of used capital for production increases, aligning with our two-period model. We observe a similar pattern for the fraction of producers in the modern sector, consumption, and output, reinforcing the notion that used capital plays a crucial role in bolstering economic activity, particularly under financial constraints.

[**Place Table 3 about here**]

Additionally, we conduct a sensitivity analysis presented in Table 4, examining variations in entry fixed costs and the productivity gap. Our findings indicate that when the fixed cost of entering the modern sector decreases, the mitigation effects of used capital on TFP loss through the extensive margin decline significantly, from 9.61% to 2.31%. The underlying intuition is that lower fixed costs enable easier entry into the modern sector, shifting the role of used capital toward reducing misallocation, which pertains to the intensive margin.

In our counterfactual experiments regarding the productivity gap, we observe that the role of used capital in enhancing TFP through the extensive margin is more pronounced when the productivity gap is low, as evidenced by increases of 6.96% (versus 3.03%) and 3.86% (versus 4.21%). This pattern arises because, when the modern sector becomes more attractive, producers tend to place less emphasis on fixed costs, aligning with the observed increase in output as the productivity gap rises from 0.5κ to 2κ.

[**Place Table 4 about here**]

4 Conclusion

In this paper, we study the novel role of the used capital market in generating aggregate productivity gain through mitigating firm-level distortions caused by financial frictions. To do so, we build general equilibrium models with heterogeneous firms, the used capital market, sectoral choices, and financial frictions in the form of collateral constraints. In our model, producers can choose between used capital and new capital. Relative to new capital, used capital has higher user cost, but is cheaper upfront and thus is easier to finance. These features make it more attractive to financially constrained firms but less attractive to unconstrained firms. In our model, financial constraints can distort producers' decisions through two channels: on the extensive margin, they distort firms' entry into the high-productivity modern sector and thus reduce the productivity of individual producers; on the intensive margin, they force firms within the modern sector to invest less than their optimal level, thus generating capital misallocation across firms.

We analytically show that allowing for the used capital market can facilitate firms' entry into the high-productivity sector and allow firms within the high-productivity sector to invest more capital and thus mitigate capital misallocation. As a result of these firm-level effects, the market for used capital can significantly reduce the aggregate productivity losses at the macro level. Quantitatively, about 9.3% productivity gain can be achieved by considering the market for used capital, and the extensive entry channel accounts for 80% of the gain. To the best of our knowledge, our paper is among the first to systemically study the role of the used capital market in mitigating firm-level distortions and in reducing aggregate productivity losses caused by financial frictions. Our results highlight the importance of accounting for the role of the used capital market in affecting micro-level firm behaviors and macro-level outcomes.

Figure 1

△ Function

In this figure, we plot the △ function in our analytical model against the net worth w. The △ function measures the value difference between the modern sector and the traditional sector.

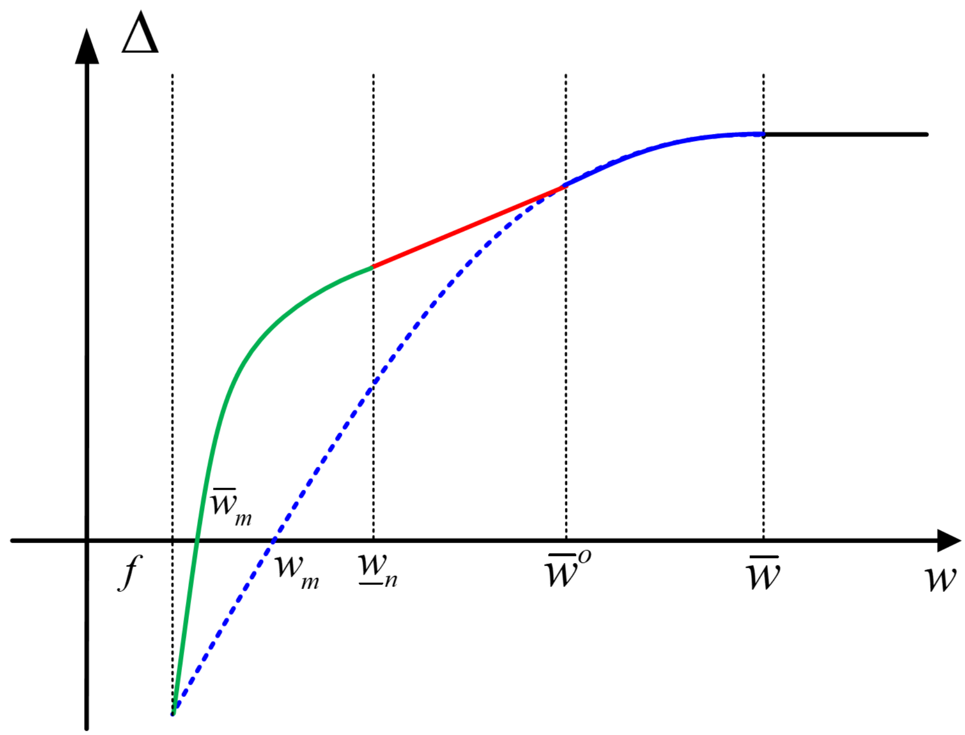

Figure 2

A Numerical Example

This figure presents the results of a numerical example to show how producers' sector choice, collateral multiplier, new capital, and used capital depend on the initial net worth w in the model economy with and without the used capital market. The top left figure illustrates producers' decision to enter the modern sector. The top right figure plots the value of the collateral constraint multiplier λ. The bottom figure shows producers' choice of new capital (on the left) and used capital (on the right), respectively. Blue lines denote the economy with used capital, and red lines denote the economy without used capital. The parameter values we use in this example are as follows: the discount factor $\beta = 0.99$, collateralizability $\theta = 0.50$, fixed cost $f = 0.10$, curvature of production $\alpha = 0.40$, the productivity gap between two sectors $\kappa = 2$, productivity $z = 1.5$, the net worth is uniform distribution on $w_{\min} = 0.01$ and $w_{\max} = 1.80$. The thresholds $\bar{w} = 1.64$, $\bar{w}^o = 1.47$, $\underline{w}_n = 1.03$, $\bar{w}_m = 0.72$, $w_m = 0.53$, the equilibrium old capital price $q = 0.5082$.

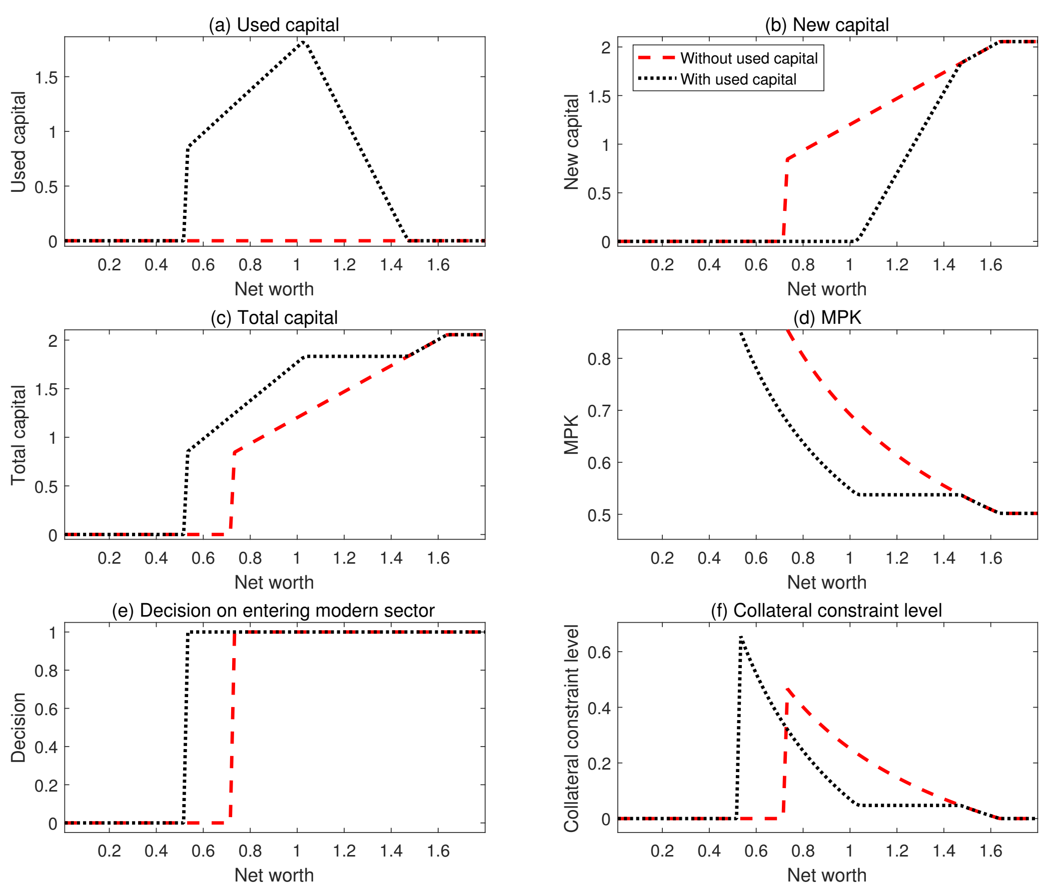

Figure 3

Decision to enter modern

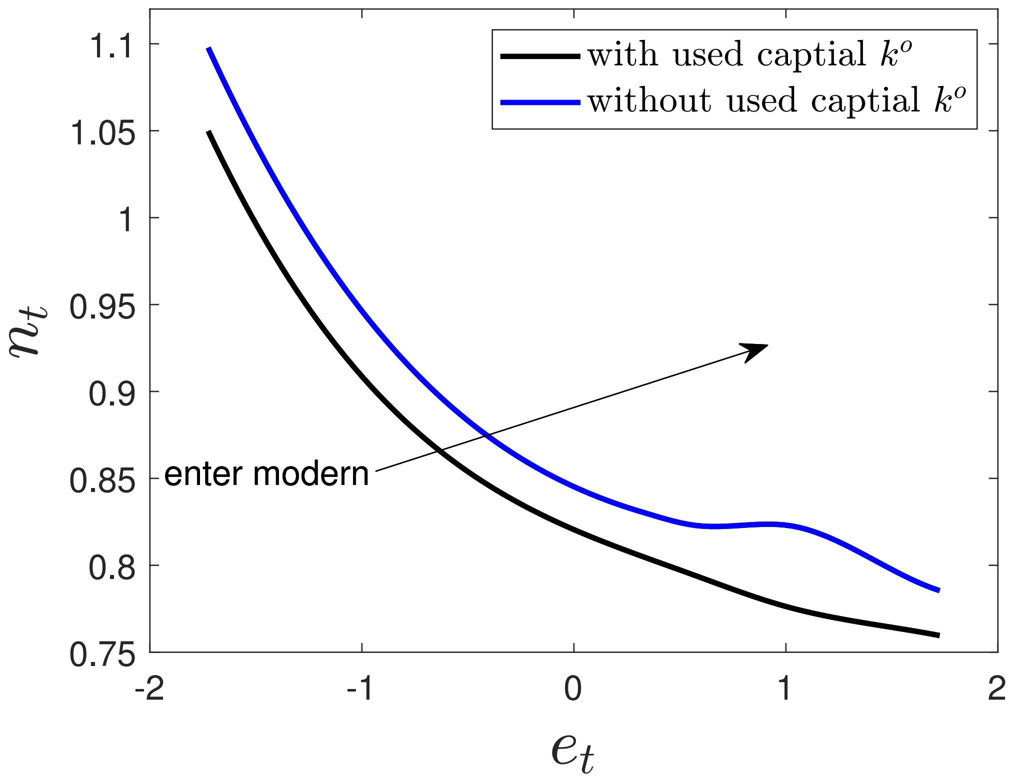

Table 1

Calibrated parameter values

This table lists the parameter values used to solve and simulate the model. We calibrate the model at the annual frequency using data moments for the U.S. economy from 1986 to 2023.

Parameters	Symbol	Value
Discount rate	β	0.94
Growth rate	γ	1.02
Capital share	α	0.33
Span of control	η	0.85
Collateralizability	θ	0.38
Fixed cost	κ	0.8
Autoregressive coefficient v	ρ_w	0.90
Innovation variance v	σ_w	0.17
A fraction of new capital becomes used capital	δ^n	0.06
A fraction of used capital becomes costless	δ^o	0.16
Productivity gap	ϕ	1.33
Persistence of idiosyncratic transitory shocks	ρ	0.50
Std. Dev. of idiosyncratic transitory shocks	σ_e	0.61
Std. Dev. of permanent shocks	σ_z	4.6

Table 2

Aggregate moments for data and model:new

This table reports key moments generated under the benchmark parameters reported in Table 1.Empirical moments are computed using U.S. annual data from 1971 to 2023. See more details of data construction in Appendix C.

Moments	Data	Model
Matched moments		
Share of investment expenditure on used capital	0.34	0.32
Investment-to-output ratio	0.22	0.17
SD of output	1.89	1.89
SD of output growth	0.32	0.37
Share of output by producers in modern sector	0.90	0.93
Unmatched moments		
Consumption-to-investment ratio	4.69	4.74
Debt to output	0.93	0.75
Real interest rate	1.41	1.00
SD of employment	1.79	1.89
SD of employment growth	0.23	0.37
SD of total capital	2.12	1.96
SD of total capital growth	0.33	0.27
1-year auto-correlation output	0.98	0.98
3-year auto-correlation output	0.96	0.96
5-year auto-correlation output	0.94	0.94
1-year auto-correlation total capital	0.98	0.99
3-year auto-correlation total capital	0.94	0.98
5-year auto-correlation total capital	0.92	0.97

Table 3

Aggregate implications for the used capital

This table summarizes the aggregate implications of our model. The "Benchmark" column reflects our baseline model, while the columns for collateralizability θ = 0.33 and θ = 0.50 represent the results from counterfactual experiments. "Yes" indicates an economy that incorporates used capital, whereas "No" denotes an economy without used capital. "The TFP loss from misallocation" quantifies the productivity loss relative to an efficient economy within the modern sector. The "Fraction of constraint" indicates the proportion of financially constrained producers operating in the modern sector. Finally, consumption and output figures encompass both the traditional and modern sectors.

	Benchmark		θ = 0.33		θ = 0.50	
	Yes	No	Yes	No	Yes	No
Debt to output (modern)	0.83	0.84	0.73	0.81	1.03	0.96
Fraction of constraint	0.96	0.97	0.99	1	0.84	0.82
Used capital ratio	0.31	0	0.32	0	0.31	0
TFP (modern)	1.16	1.06	1.15	0.95	1.17	1.12
TFP loss from misallocation, %	2.83	4.40	2.97	5.03	2.42	3.43
TFP attain of used capital from misallocation, %	1.65		2.37		1.02	
TFP attain of used capital from entry,%	7.63		18.6		3.33	
Fraction producers (modern)	0.65	0.39	0.62	0.20	0.68	0.55
Fraction output (modern)	0.93	0.75	0.92	0.53	0.94	0.86
Consumption	1.75	0.92	1.69	0.72	1.85	1.12
Output	2.20	1.12	2.10	0.82	2.38	1.42

Table 4

Sensitivity analysis

The table shows the sensitivity analysis about fixed cost f and productivity gap κ. The other details are same in Table 3.

	Benchmark		$f^n = 0.5f$		$f^n = 2f$	
	Yes	No	Yes	No	Yes	No
Debt to output (modern)	1.45	1.68	1.21	1.42	1.61	1.72
Fraction of constraint	0.71	0.65	0.65	0.71	0.78	0.81
Used capital ratio	0.35	0	0.31	0	0.42	0
TFP (modern)	2.15	1.89	2.24	1.86	1.86	1.72
TFP loss from misallocation, %	2.21	4.26	2.32	3.96	2.02	5.69
TFP attain of used capital from misallocation, %	3.12		3.06		5.34	
TFP attain of used capital from entry,%	5.21		2.31		9.61	
Fraction producers (modern)	0.72	0.54	0.79	0.65	0.65	0.59
Fraction output (modern)	0.85	0.76	0.86	0.81	0.65	0.61
Consumption	0.98	0.91	1.06	1.01	0.92	0.86
Output	1.68	1.21	1.86	1.56	1.56	1.48
	Benchmark		$\kappa^n = exp(0.5\kappa)$		$\kappa^n = exp(2\kappa)$	
	Yes	No	Yes	No	Yes	No
Debt to output (modern)	1.45	1.68	1.61	1.84	1.41	1.62
Fraction of constraint	0.71	0.65	0.75	0.85	0.68	0.79
Used capital ratio	0.35	0	0.40	0	0.36	0
TFP (modern)	2.15	1.89	2.01	1.96	2.64	2.12
TFP loss from misallocation, %	2.21	4.26	1.86	2.31	2.01	2.96
TFP attain of used capital from misallocation, %	3.12		3.03		4.21	
TFP attain of used capital from entry,%	5.21		6.96		3.86	
Fraction producers (modern)	0.72	0.54	0.65	0.59	0.86	0.71
Fraction output (modern)	0.85	0.76	0.76	0.65	0.91	0.82
Consumption	0.98	0.91	0.85	0.79	1.32	1.12
Output	1.68	1.21	1.56	1.21	1.81	1.73

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Internet Appendix

A Appendix to the analytical model

A.1 Derivations of the optimization problems

A.1.1 Producers in modern sector

For producers already choosing to enter the modern sector, we denote the multipliers on the budget constraints by $\mu_{0,t}$ and $\beta\mu_{1,t+1}$, the collateral constraint by $\beta\lambda_t$, and on non-negativity constraint for new and old capital by ν_t^n and ν_t^o, respectively. Then, we can the Lagrangian function is

$$
\begin{aligned}
L &= \max_{C_{1,t+1}^{mod}, K_t^n, K_t^o, B_t^{mod}} && \beta C_{1,t+1}^{mod} \\
&+ \mu_{0,t} && \left[B_t^{mod} + w_{0,t} - K_t^n - q_t K_t^o - f\right] \\
&+ \beta\mu_{1,t+1} && \left[Y_{t+1}^{mod} - R_t B_t^{mod} + q_{t+1} K_t^n - C_{1,t+1}^{mod}\right], \\
&+ \beta\lambda_t && \left[q_{t+1}\theta K_t^n - R_t B_t^{mod}\right] \\
&+ v_t^n K_t^n && \\
&+ v_t^o K_t^o &&
\end{aligned} \tag{A.1}
$$

We can derive following first order conditions w.r.t K_t^n, K_t^o, B_t^{mod}, and $C_{1,t+1}^{mod}$, respectively:

$$\beta\mu_{1,t}\left[\frac{\partial Y_{t+1}^{mod}}{\partial K_t^n} + q_{t+1}\right] + \beta\lambda_t\theta q_{t+1} + v_t^n = \mu_{0,t} \tag{A.2}$$

$$\beta\mu_{1,t+1}\left[\frac{\partial Y_{t+1}^{mod}}{\partial K_t^o}\right] + v_t^o = \mu_{0,t} q_t \tag{A.3}$$

$$\mu_{0,t} = \mu_{1,t+1} + \lambda_t \tag{A.4}$$

$$\mu_{1,t} = 1 \tag{A.5}$$

The marginal value of net worth at date t is $\mu_{0,t} = 1 + \lambda_t \geq 1$.This reflects the additional value due to the collateral constraint. In contrast, the producer's marginal value of net wealth at time $t+1$ is $\mu_{1,t+1} = 1$, as it consumes all its remaining positive net wealth.

A.1.2 Producers in traditional sector

For producers who choose to stay in the traditional sector, we denote the multipliers on the budget constraints by $\mu_{0,t}^{tra}$ and $\beta\mu_{1,t+1}^{tra}$, respectively. The Lagrangian can be written as

$$\begin{aligned} L \;=\; & \max_{C_{1,t+1}^{tra}, B_t^{tra}} \beta C_{1,t+1}^{\mathrm{tr}\,a} \\ & +\mu_{0,t}^{tra} \left[B_t^{tra} + w_{0t}\right] \\ & +\beta\mu_{1,t+1}^{tra} \left[Y_{t+1}^{tra} - R_t B_t^{tra} - C_{1,t+1}^{tra}\right] \end{aligned} \quad . \tag{A.6}$$

We can derive following first order conditions w.r.t B_{t+1}^{tra} and $C_{1,t+1}^{tra}$, respectively:

$$\beta = \beta\mu_{1,t+1}^{tra} \Rightarrow \mu_{1,t+1}^{tra} = 1 \tag{A.7}$$

$$\mu_{0,t}^{tra} = \beta\mu_{1,t+1}^{tra} R_t \Rightarrow \mu_{0,t}^{tra} = 1 \tag{A.8}$$

For producers in the traditional sector, they can not borrow, their marginal value of net worth is 1 when they are young and old.

A.2 Proof of proposition 1

We are interested in considering the case in which both types of capital are used in equilibrium, that is, the case in which neither type of capital is dominated.

1. In equilibrium, the user cost of new capital for an unconstrained firm has to be less than or equal to the user cost of used capital, that is $u_n \leq u_o$, which implies that $1 - \beta q \leq q$, or equivalently $q \geq \frac{1}{1+\beta}$.

   - **Proof by contradiction**: if $u_n > u_o$, then new capital will be strictly dominated since we have $\psi_n > u_n > u_o = \psi_o$, then we have

     $$u_n + \frac{\lambda}{\mu_1}\psi_n > u_o + \frac{\lambda}{\mu_1}\psi_o \geq \beta\frac{\partial Y_t^m}{\partial k_t}$$

     which implies that $\nu_n > 0$, or $k_n = 0$, so an unconstrained firm will not choose new capital, which is not an equilibrium.

2. In equilibrium, the down payment on new capital has to strictly exceed the down payment on used capital, that is $\psi_n > \psi_o$.

- **Proof by contradiction**: if $\psi_n \leq \psi_o$ instead, given we have shown that $u_n \leq u_o$, then we have $u_o = \psi_o \geq \psi_n > u_n$, but then there will no investment in used capital, which is not an equilibrium either, because we have

$$u_o + \frac{\lambda}{\mu_1}\psi_o \geq u_n + \frac{\lambda}{\mu_1}\psi_n \geq \beta\frac{\partial Y_t^m}{\partial k_t}$$

which implies that $\nu_o > 0$, or $k_o = 0$. Then, from $\psi_n > \psi_o$, we have $1 - \beta\theta q > q$, or equivalently $q < \frac{1}{1+\beta\theta}$.

- The price of used capital satisfies $\frac{1}{1+\beta} \leq q < \frac{1}{1+\beta\theta}$ in equilibrium. Taking the above conditions together, we have

$$\frac{1}{1+\beta} \leq q < \frac{1}{1+\beta\theta}$$

and the actual value of q is determined by the market clearing condition of used capital, or depending on who is the marginal buyer that is indifferent between old capital and new capital.

A.3 Proof of proposition 2

To prove Proposition 2, we first use the optimality conditions of producers in the modern sector derived in Appendix A.1.1 to define the user cost of used and new capital for an arbitrage producer with initial net worth w. As we have shown in Section 2.2.2, in the presence of financial friction, the user cost of new capital depends on the net worth w. Then, we characterize producers' choice between new and used capital by comparing their user costs. If the user costs of two types of capital are different, the producer will choose the one with lower user cost; while if the user costs of two types of capital are equal, the producer will use both types of capital.

Given the optimality conditions of producers in the modern sector derived in Appendix A.1.1, in the stationary equilibrium, for a producer with net worth w, its user cost of new capital u_t^{nn} and old capital u_t^{oo} in terms of consumption goods at time t is defined as:

$$\begin{aligned} u^{nn}(w) =& 1 - \beta\frac{\mu_1}{\mu_0\,(w)}q - \beta\frac{\mu_1}{\mu_0\,(w)}\lambda(w)\theta q, \\ u^{oo}(w) =& q \end{aligned}$$

where $\beta \frac{\mu_1}{\mu_0(w)} = \beta \frac{1}{1+\lambda(w)}$ is the discount factor. The user cost of new capital is equal to the current price of new capital, 1, minus the discounted resale value, $\frac{\beta q}{1+\lambda(w)}$ and the marginal value of relaxing the collateral constraint for owning this capital. The user cost of used capital is equal to its price q since it has no resale value in the next period and thus can not be used for borrowing.

For ease of comparison, we multiply the above user costs by $1 + \lambda(w)$ and relabel them as:

$$u^n(w) = 1 - \beta q + \lambda(w)(1 - \beta\theta q), \tag{A.9}$$
$$u^o(w) = q + \lambda(w)q. \tag{A.10}$$

In the following proof, we directly compare $u^n(w)$ and $u^o(w)$ to characterize producers' choice between used capital and new capital. There are a total of 3 possible relationships between $u^n(w)$ and $u^o(w)$, we discuss them one by one.

1. If $u^n(w) - u^o(w) = 0$, meaning that the user cost of new capital and used capital are equalized, then the producer with net worth w will use both used capital and new capital. In this case, the value of the Lagrangian multiplier

$$\bar{\lambda} = \frac{1 - \beta q - q}{q(1+\beta\theta) - 1}$$

Using the producer's budget constraint (equation (10)) and the definition of user cost in equation (A.9), we can derive that:

$$u^o(w) = \left(1 + \bar{\lambda}\right) q = q + \frac{1 - \beta q - q}{q(1+\beta\theta) - 1} q = \beta \left[\alpha \kappa^{1-\alpha} z^{1-\alpha} \left(K^o + K^n\right)^{\alpha - 1}\right] \tag{A.11}$$

In this case, the total amount of capital K is constant and is given by

$$K = \left(\frac{q\left(1 + \frac{1-\beta q - q}{q(1+\beta\theta)-1}\right)}{\beta\alpha\left(\kappa z\right)^{1-\alpha}} \right)^{\frac{1}{\alpha-1}}. \tag{A.12}$$

The producer uses its net worth w to pay for the cost of used capital, the total down payment of new capital, and the fixed entry cost,

$$w = qK^o + (1 - \beta\theta q)K^n + f. \tag{A.13}$$

Combining equation (A.12) and (A.13), we can further solve the value of K^o and K^n as

$$K^n = \frac{-q}{1-\beta q\theta-q}\left(\frac{q+\frac{1-\beta q-q}{q(1+\beta\theta)-1}q}{\beta\alpha\kappa^{1-\alpha}z^{1-\alpha}}\right)^{\frac{1}{\alpha-1}} + \frac{w-f}{1-\beta q\theta-q}, \tag{A.14}$$

$$K^o = \frac{1-\beta q\theta}{1-\beta q\theta-q}\left(\frac{q+\frac{1-\beta q-q}{q(1+\beta\theta)-1}q}{\beta\alpha\kappa^{1-\alpha}z^{1-\alpha}}\right)^{\frac{1}{\alpha-1}} - \frac{w-f}{1-\beta q\theta-q}. \tag{A.15}$$

Moreover, we define the upper bound of net worth for these producers as $\bar{w}^o$ (if a producer's net worth w is above this value, it will not invest in used capital anymore), it satisfies:

$$q + \frac{1-\beta q-q}{q(1+\beta\theta)-1}q = \beta\left[\alpha\kappa^{1-\alpha}z^{1-\alpha}\left(\frac{\bar{w}^o-f}{1-\beta\theta q}\right)^{\alpha-1}\right] \tag{A.16}$$

Similarly, we can also define the lower bound of net worth for these producers as $\underline{w}_n$ (if a producer's net worth w is below this value, it will not use new capital anymore), it satisfies the following condition:

$$q + \frac{1-\beta q-q}{q(1+\beta\theta)-1}q = \beta\left[\alpha\kappa^{1-\alpha}z^{1-\alpha}\left(\frac{\underline{w}_n-f}{q}\right)^{\alpha-1}\right] \tag{A.17}$$

When the producer's net worth falls into the range $(\underline{w}_n, \bar{w}^o)$, the producer will use both types of capital. From equation (A.14) and (A.15), it is clear that as the producer's net worth increases from $\underline{w}_n$ to $\bar{w}^o$, it will increase its investment in new capital while decreasing its investment in used capital.

2. If $u^n(w) - u^o(w) > 0$, meaning that the user cost of new capital is higher than that of used capital for the producer, then the producer will only invest in used capital. According to definition of $u^n(w)$ and $u^o(w)$ in equation (A.9) and (A.10), $u^n(w) - u^o(w) > 0$ is equivalent to $\lambda > \bar{\lambda}$, which is true only when $w < \underline{w}_n$. In this case, the producer invests only in used capital, so we have

$$K^o = \frac{w-f}{q}; K^n = 0.$$

3. $u^n(w) - u^o(w) < 0$, meaning that the user cost of new capital is lower than that of used capital for the producer, the producer will only use new capital. According to definition of $u^n(w)$ and $u^o(w)$ in equation (A.9) and (A.10), $u^n(w) - u^o(w) < 0$ is

equivalent to $\lambda < \bar{\lambda}$, which is true only when $w > \underline{w}_o$. In this case, the producer invests only in new capital. If the firm is still constrained, we have

$$u^n(w) = 1 - \beta q + \lambda(1 - \beta\theta q) = \beta \left[\alpha\kappa^{1-\alpha} z^{1-\alpha} \left(K^n\right)^{\alpha-1} \right]$$

and the total amount of new capital the producer can purchase is

$$K^n = \frac{w - f}{1 - \beta\theta q}; K^o = 0.$$

We can then define smallest required net worth $\bar{w}$ for firms to be unconstrained (thus $\lambda = 0$) using following equation

$$1 - \beta q = \beta \left[\alpha\kappa^{1-\alpha} z^{1-\alpha} \left(\frac{\bar{w} - f}{1 - \beta\theta q} \right)^{\alpha-1} \right]. \tag{A.18}$$

Combining equation (A.16) and (A.18), it is clear that: 1) if $q > \frac{1}{1+\beta}$, $q + \frac{1-\beta q - q}{q(1+\beta\theta)-1} q > 1 - \beta q$, we must have $\bar{w}^o < \bar{w}$; 2) while if $q = \frac{1}{1+\beta}$, $q + \frac{1-\beta q - q}{q(1+\beta\theta)-1} q = 1 - \beta q$, we must have $\bar{w}^o = \bar{w}$.

Finally, if the producer's net worth is even higher than $\bar{w}$, it will achieve its optimal investment scale with the total amount of capital as follows:

$$K = \left(\frac{1 - \beta q}{\beta\alpha \left(\kappa z\right)^{1-\alpha}} \right)^{\frac{1}{\alpha-1}}.$$

The producer consumes extra net worth $w - \bar{w}$.

A.4 Time 1 consumption C_1^{mod} and $\triangle$ function

A.4.1 Time 1 consumption C_1^{mod}

Depending on the level of producers' net worth, there are a total of 4 cases to consider if the producer chooses the modern sector:

1. With net worth $w \leq \underline{w}_n$, the producer is sufficiently constrained and only invests in used capital, its C_1^{mod} can be calculated as

$$C_1^{mod,o} = \kappa^{1-\alpha} z^{1-\alpha} \left(\frac{w - f}{q} \right)^{\alpha}.$$

2. With net worth $w \in (\underline{w}_n, \bar{w}^o)$, the producer is constrained but invests in both used capital and new capital, its C_1^{mod} is given by

$$C_1^{mod,on} = \left((\kappa z)^{1-\alpha} + \frac{q^2\theta - q^2}{1 - \beta q\theta - q} \right) \left(\frac{q + \frac{1-\beta q - q}{q(1+\beta\theta)-1} q}{\beta\alpha (\kappa z)^{1-\alpha}} \right)^{\frac{1}{\alpha-1}} + \frac{q - q\theta}{1 - \beta q\theta - q} (w - f) .$$

3. With net worth $w \in [\bar{w}^o, \bar{w})$, the producer is constrained and invests only in new capital, then C_1^{mod} is given by

$$C_1^{mod,nc} = (\kappa z)^{1-\alpha} \left(\frac{w - f}{1 - \beta\theta q} \right)^{\alpha} - (\beta^{-1} - q) \frac{w - f}{1 - \beta\theta q} - \beta^{-1} f + \beta^{-1} w.$$

4. With net worth $w \geq \bar{w}$, the producer is unconstrained and only invests in new capital, C_1^{mod} is given by

$$C_1^{mod,nu} = (\kappa z)^{1-\alpha} \left(\frac{1 - \beta q}{\beta\alpha\kappa^{1-\alpha} z^{1-\alpha}} \right)^{\frac{\alpha}{\alpha-1}} - (\beta^{-1} - q) \left(\frac{1 - \beta q}{\beta\alpha (\kappa z)^{1-\alpha}} \right)^{\frac{1}{\alpha-1}} - \beta^{-1} f + \beta^{-1} w.$$

A.4.2 Derivation of the $\triangle$ function

There are a total of 4 cases to consider if the producer chooses the modern sector:

1. With net worth $w \leq \underline{w}_n$, the producer is sufficiently constrained and only invests in used capital, $\triangle_o = C_1^{mod,o} - C_1^{tra}$ can be calculated as

$$\triangle_o = \kappa^{1-\alpha} z^{1-\alpha} \left(\frac{w - f}{q} \right)^{\alpha} - z^{1-\alpha} - \beta^{-1} w$$

2. With net worth $w \in (\underline{w}_n, \bar{w}^o)$, the producer is constrained but invests in both used

capital and new capital, $\triangle_{on} = C_1^{mod,on} - C_1^{tra}$ is given by

$$\begin{aligned}
\triangle_{on} &= \left((\kappa z)^{1-\alpha} + \frac{q^2\theta - q^2}{1-\beta q\theta - q} \right) \left(\frac{q + \frac{1-\beta q - q}{q(1+\beta\theta)-1} q}{\beta\alpha (\kappa z)^{1-\alpha}} \right)^{\frac{1}{\alpha-1}} + \frac{q - q\theta}{1-\beta q\theta - q} (w - f) - z^{1-\alpha} - \beta^{-1} w \\
&= \left((\kappa z)^{1-\alpha} + \frac{q^2\theta - q^2}{1-\beta q\theta - q} \right) \left(\frac{q + \frac{1-\beta q - q}{q(1+\beta\theta)-1} q}{\beta\alpha (\kappa z)^{1-\alpha}} \right)^{\frac{1}{\alpha-1}} + \left(\frac{q - q\theta}{1-\beta q\theta - q} - \beta^{-1} \right) w \\
&\quad - \frac{q - q\theta}{1-\beta q\theta - q} f - z^{1-\alpha}
\end{aligned}$$

Note that $\left(\frac{q-q\theta}{1-\beta q\theta - q} - \beta^{-1} \right) = \frac{q\left(1+\beta^{-1}\right) - \beta^{-1}}{1-\beta q\theta - q} \geq 0$ since $\frac{1}{1+\beta} \leq q < \frac{1}{1+\beta\theta}$, so it follows that $\triangle_{on}$ is increasing in inital net worth w.

3. With net worth $w \in [\bar{w}^o, \bar{w})$, the producer is constrained and invests only in new capital, then $\triangle_{nc} = C_1^{mod,nc} - C_1^{tra}$ is given by

$$\begin{aligned}
\triangle_{nc} &= (\kappa z)^{1-\alpha} \left(\frac{w-f}{1-\beta\theta q} \right)^{\alpha} - (\beta^{-1} - q) \frac{w-f}{1-\beta\theta q} - \beta^{-1} f + \beta^{-1} w - z^{1-\alpha} - \beta^{-1} w \\
&= (\kappa z)^{1-\alpha} \left(\frac{w-f}{1-\beta\theta q} \right)^{\alpha} - (\beta^{-1} - q) \frac{w-f}{1-\beta\theta q} - \beta^{-1} f - z^{1-\alpha}
\end{aligned}$$

4. With net worth $w \geq \bar{w}$, the producer is unconstrained and only invests in new capital, then $\triangle_{nu} = C_1^{mod,nu} - C_1^{tra}$ is given by

$$\begin{aligned}
\triangle_{nu} &= (\kappa z)^{1-\alpha} \left(\frac{1-\beta q}{\beta\alpha (\kappa z)^{1-\alpha}} \right)^{\frac{\alpha}{\alpha-1}} - (\beta^{-1} - q) \left(\frac{1-\beta q}{\beta\alpha\kappa^{1-\alpha} z^{1-\alpha}} \right)^{\frac{1}{\alpha-1}} - \beta^{-1} f + \beta^{-1} w - z^{1-\alpha} - \beta^{-1} w \\
&= (\kappa z)^{1-\alpha} \left(\frac{1-\beta q}{\beta\alpha (\kappa z)^{1-\alpha}} \right)^{\frac{\alpha}{\alpha-1}} - (\beta^{-1} - q) \left(\frac{1-\beta q}{\beta\alpha (\kappa z)^{1-\alpha}} \right)^{\frac{1}{\alpha-1}} - \beta^{-1} f - z^{1-\alpha}
\end{aligned}$$

We summarize the above 4 cases in the following equation:

$$\triangle = \begin{cases} \triangle_o = C_1^{mod,o} - C_1^{tra} & w \leq \underline{w}_n \\ \triangle_{on} = C_1^{mod,on} - C_1^{tra} & w \in (\underline{w}_n, \bar{w}^o) \\ \triangle_{nc} = C_1^{mod,nc} - C_1^{tra} & w \in [\bar{w}^o, \bar{w}) \\ \triangle_{nu} = C_1^{mod,nu} - C_1^{tra} & w \geq \bar{w} \end{cases} .$$

A.4.3 Shape of the $\triangle$ function

To show how the $\triangle$ function changes with the initial net worth w, we calculate the derivative of each part of $\triangle$ function with respect to the initial net worth w.

1. For $\triangle_o$, when $w \le \underline{w}_n$, take derivative with respect to w, we can obtain

$$\frac{\partial \triangle_o}{\partial w} = \alpha\,(\kappa z)^{1-\alpha}\left(\frac{w-f}{q}\right)^{\alpha-1}\frac{1}{q} - \beta^{-1} \tag{A.19}$$

   Given $\alpha < 1$, $\frac{\partial \triangle_o}{\partial w}$ is decreasing in w for $w \in [w_{min}, \underline{w}_n]$. Since $w \le \underline{w}_n$, so we have

$$\frac{\partial \triangle_o}{\partial w} \ge \left[\frac{\partial \triangle_o}{\partial w}|\underline{w}_n\right] = \alpha\kappa^{1-\alpha}z^{1-\alpha}\left(\frac{\underline{w}_n - f}{q}\right)^{\alpha-1}\frac{1}{q} - \beta^{-1}$$

   From the definition of $\underline{w}_n$ in equation (A.17), we have

$$\begin{aligned}\frac{\partial \triangle_o}{\partial w} \ge \left[\frac{\partial \triangle_o}{\partial w}|\underline{w}_n\right] &= \alpha\kappa^{1-\alpha}z^{1-\alpha}\left(\frac{\underline{w}_n - f}{q}\right)^{\alpha-1}\frac{1}{q} - \beta^{-1}\\ &= \left[q + \frac{1-\beta q - q}{q(1+\beta\theta)-1}q\right]\frac{1}{\beta}\frac{1}{q} - \beta^{-1}\\ &= \left[\frac{1-\beta q - q}{q(1+\beta\theta)-1}\right]\beta^{-1}\\ &\ge 0\end{aligned}$$

   since we have $\frac{1}{1+\beta} \le q < \frac{1}{1+\beta\theta}$, it follows that $1-\beta q - q = 1-(1+\beta)q \le 0$, $q(1+\beta\theta)-1 < 0$, and thus $\left[\frac{1-\beta q-q}{q(1+\beta\theta)-1}\right] \ge 0$.

   To summarize, $\triangle_o$ is an increasing and concave function in $w \in [w_{min}, \underline{w}_n]$.

2. For $\triangle_{on}$, when $w \in (\underline{w}_n, \bar{w}^o)$, taking derivative with respect to w, we can obtain a non-negative constant slop,

$$\begin{aligned}\frac{\partial \triangle_{on}}{\partial w} &= \left(\frac{q-q\theta}{1-\beta q\theta - q} - \beta^{-1}\right)\\ &= \left[\frac{1-\beta q - q}{q(1+\beta\theta)-1}\right]\beta^{-1}\\ &= \left[\frac{\partial \triangle_o}{\partial w}|\underline{w}_n\right]\\ &\ge 0.\end{aligned}$$

3. For $\triangle_{nc}$, when $w \in [\bar{w}^o, \bar{w})$, taking derivative with respect to w, we can obtain

$$\frac{\partial \triangle_{nc}}{\partial w} = \alpha \kappa^{1-\alpha} z^{1-\alpha} \left(\frac{w-f}{1-\beta\theta q} \right)^{\alpha-1} \frac{1}{1-\beta\theta q} - \frac{\beta^{-1}-q}{1-\beta\theta q}. \tag{A.20}$$

Again, since $\alpha < 1$, $\frac{\partial \triangle_{nc}}{\partial w}$ is decreasing in w. For $w \in [\bar{w}^o, \bar{w})$, we have

$$\left[\frac{\partial \triangle_{nc}}{\partial w}|\bar{w}\right] < \frac{\partial \triangle_{nc}}{\partial w} \leq \left[\frac{\partial \triangle_{nc}}{\partial w}|\bar{w}^o\right]$$

According to the definition of $\bar{w}^o$ in equation (A.16), we have

$$\begin{aligned}
\left[\frac{\partial \triangle_{nc}}{\partial w}|\bar{w}^o\right] &= \left[q + \frac{1-\beta q - q}{q(1+\beta\theta)-1} q\right] \frac{1}{\beta} \frac{1}{1-\beta\theta q} - \frac{\beta^{-1}-q}{1-\beta\theta q} \\
&= \frac{1-q-\beta q}{q(1+\beta\theta)-1} \frac{1}{\beta} \\
&\geq 0
\end{aligned}$$

Again, since $\frac{1}{1+\beta} \leq q < \frac{1}{1+\beta\theta}$, it follows that $1-\beta q - q = 1-(1+\beta)q \leq 0$, $q(1+\beta\theta) - 1 < 0$, and $\left[\frac{1-\beta q - q}{q(1+\beta\theta)-1}\right] \geq 0$.

Additionally, according to the definition of $\bar{w}$ in equation (A.18), we have

$$\left[\frac{\partial \triangle_{nc}}{\partial w}|\bar{w}\right] = (1-\beta q) \frac{1}{\beta} \frac{1}{1-\beta\theta q} - \frac{\beta^{-1}-q}{1-\beta\theta q} = 0$$

As a result, $\triangle_{nc}$ is increasing and concave for $w \in [\bar{w}^o, \bar{w})$. It reaches the highest value at $\bar{w}$.

4. For $\triangle_{nu}$, which is independent of w, we have

$$\frac{\partial \triangle_{nu}}{\partial w} = 0. \tag{A.21}$$

In summary, we have the following conditions:

$$\frac{\partial \Delta}{\partial w} = \begin{cases}
\frac{\partial \triangle_o}{\partial w} \geq \left[\frac{1-\beta q - q}{q(1+\beta\theta)-1}\right] \beta^{-1} & w \leq \underline{w}_n \\
\frac{\partial \triangle_{on}}{\partial w} = \left[\frac{1-\beta q - q}{q(1+\beta\theta)-1}\right] \beta^{-1} & w \in (\underline{w}_n, \bar{w}^o) \\
0 < \frac{\partial \triangle_{nc}}{\partial w} \leq \left[\frac{1-\beta q - q}{q(1+\beta\theta)-1}\right] \beta^{-1} & w \in [\bar{w}^o, \bar{w}) \\
\frac{\partial \triangle_{nu}}{\partial w} = 0 & w \geq \bar{w}
\end{cases}.$$

Given that $\left[\frac{\partial\triangle_o}{\partial w}|w=\underline{w}_n\right]=\frac{\partial\triangle_{on}}{\partial w}=\left[\frac{\partial\triangle_{nc}}{\partial w}|w=\bar{w}^o\right]$, it follows that $\triangle_{on}$ is a tangent line to both $\triangle_o$ and $\triangle_{nc}$, and $\underline{w}_n$ and $\bar{w}^o$ are the x-coordinate of the tangent points.

A.5 Assumption 1

First, we must ensure that there always exist producers in the traditional sector. This requires that producers with the lowest net worth must choose the traditional sector. In other words, they are unable to pay for the fixed cost using their net worth, which leads to

$$f > w_{min}. \tag{A.22}$$

Second, we must ensure that producers with sufficient net worth will choose the modern sector. This requires $\triangle_{nu}$ (defined in Appendix A.4) to be positive for producers with $w \geq \bar{w}$, which implies

$$(\kappa z)^{1-\alpha}\left(\frac{1-\beta q}{\beta\alpha\left(\kappa z\right)^{1-\alpha}}\right)^{\frac{\alpha}{\alpha-1}}-(\beta^{-1}-q)\left(\frac{1-\beta q}{\beta\alpha\left(\kappa z\right)^{1-\alpha}}\right)^{\frac{1}{\alpha-1}}-\beta^{-1}f+\beta^{-1}w > z^{1-\alpha}+\beta^{-1}w.$$

Given the value of z and the price of used capital q, we must have

$$f < \bar{f}_{max}=\beta\left[(\kappa z)^{1-\alpha}\left(\frac{1-\beta q}{\beta\alpha\left(\kappa z\right)^{1-\alpha}}\right)^{\frac{\alpha}{\alpha-1}}-(\beta^{-1}-q)\left(\frac{1-\beta q}{\beta\alpha\left(\kappa z\right)^{1-\alpha}}\right)^{\frac{1}{\alpha-1}}-z^{1-\alpha}\right] \tag{A.23}$$

Finally, we make a technical assumption to ensure that w_m defined in equation A.27 is lower than $\underline{w}_n$ defined in equation (A.17). This assumption facilitates the comparison of TFP in Section 4. Specifically, to ensure $w_m < \underline{w}_n$, we must have $\triangle_{nc}\left(\underline{w}_n\right) > 0$ since $\triangle_{nc}$ is an increasing function of w and $\triangle_{nc}\left(w_m\right)=0$. This requires,

$$\kappa^{1-\alpha}z^{1-\alpha}\left(\frac{\underline{w}_n-f}{1-\beta\theta q}\right)^{\alpha}-(\beta^{-1}-q)\frac{\underline{w}_n-f}{1-\beta\theta q}-\beta^{-1}f-z^{1-\alpha}>0. \tag{A.24}$$

Combining with the definition of $\underline{w}_n$ in equation (A.17), we can derive another value for f such that

$$f<\tilde{f}_{max}=\beta\left[\left[\frac{q+\frac{1-\beta q-q}{q(1+\beta\theta)-1}q}{\beta\alpha}\right]^{\frac{\alpha}{\alpha-1}}kz\left(\frac{q}{1-\beta\theta q}\right)^{\alpha}-(\beta^{-1}-q)\left[\frac{q+\frac{1-\beta q-q}{q(1+\beta\theta)-1}q}{\beta\alpha}\right]^{\frac{1}{\alpha-1}}\kappa z\frac{q}{1-\beta\theta q}-z^{1-\alpha}\right].$$

Then, we can further define a f_{max},

$$f_{max} = \min\left[\bar{f}_{max}, \tilde{f}_{max}\right], \tag{A.25}$$

such that the above two conditions are both satisfied.

A.6 Proof of proposition 3

A.6.1 With used capital market

According to the derivation in Appendix A.4.3, for $w \in [w_{min}, \underline{w}_n]$, $\triangle_o$ is an increasing concave function. Based on these results, we can define a cutoff value $\bar{w}_m$ such that:

$$\kappa^{1-\alpha} z^{1-\alpha} \left(\frac{\bar{w}_m - f}{q}\right)^{\alpha} - z^{1-\alpha} - \beta^{-1}\bar{w}_m = 0 \tag{A.26}$$

Additionally, from the expression of $\triangle_o$ in Appendix A.4.2, it is easy to check that

$$\Delta_o(f) = \kappa^{1-\alpha} z^{1-\alpha} \left(\frac{f - f}{q}\right)^{\alpha} - z^{1-\alpha} - \beta^{-1} f = -\beta^{-1} f - z^{1-\alpha} < 0.$$

Given $\triangle_o$ is an increasing concave function for $w \in [w_{min}, \underline{w}_n]$, and $\Delta_o(f) < 0 = \Delta_o(\bar{w}_m)$, so we confirm that $f < \bar{w}_m$.

As a result, producers with $w < \bar{w}_m$ choose the traditional sector, while producers with $w \in [\bar{w}_m, w_{max}]$ choose the modern sector.

A.6.2 Without used capital market

If the used capital market is closed, producers can only invest in new capital, then the value difference will be the same as $\triangle_{nc}$ derived in Appendix A.4.2 when $w < \bar{w}^o$. If $\triangle_{nc} < 0$, producers will choose the traditional sector. Given we have shown that $\triangle_{nc}$ is increasing and concave in Appendix A.4.3, we can define the threshold w_m such that

$$\kappa^{1-\alpha} z^{1-\alpha} \left(\frac{w_m - f}{1 - \beta\theta q}\right)^{\alpha} - (\beta^{-1} - q)\frac{w_m - f}{1 - \beta\theta q} - \beta^{-1} f - z^{1-\alpha} = 0. \tag{A.27}$$

As a result, if $w < w_m$, producers choose the traditional sector; if $w \geq w_m$ producers will choose the modern sector. Additionally, if $w \in [w_m, \bar{w}]$, same as we derived in Proposition 2, producers are still constrained; if $w \geq \bar{w}$, producers become unconstrained.

A.6.3 Comparison of the thresholds

To prove the relationships of threshold values in Proposition 3, we proceed in the following steps:

1. As we have shown in Appendix Section A.6.1, $f < \bar{w}_m$.

2. Given the definition of $\bar{w}_m$ in equation (A.26) and the definition of w_m in equation (A.27), we would like to show that $\bar{w}_m < w_m$. Starting from the expressions of Δ_{nc} and Δ_o derived in Appendix Section A.4.2, it is easy to see that

$$\Delta_{nc}\left(f\right) = \kappa^{1-\alpha} z^{1-\alpha} \left(\frac{f-f}{1-\beta\theta q}\right)^{\alpha} - (\beta^{-1} - q)\frac{f-f}{1-\beta\theta q} - \beta^{-1} f - z^{1-\alpha} = -\beta^{-1} f - z^{1-\alpha} < 0$$

   Since Δ_{nc} is increasing when $w < \bar{w}$, and $\Delta_{nc}\left(f\right) < \Delta_{nc}\left(w_m\right) = 0$, so we must have $f < w_m$. Moreover, from Δ_o, we also know that

$$\Delta_o\left(f\right) = \kappa^{1-\alpha} z^{1-\alpha} \left(\frac{f-f}{q}\right)^{\alpha} - z^{1-\alpha} - \beta^{-1} f = 0 = -\beta^{-1} f - z^{1-\alpha} < 0$$

   Therefore, we have $f < \bar{w}_m$. Notice that,

$$\Delta_o\left(f\right) = \Delta_{nc}\left(f\right) < 0$$

   and both Δ_o and Δ_{nc} are monotonic increasing and concave. By the condition in Assumption 1, $\Delta_o(\underline{w}_n) > \Delta_{nc}(\underline{w}_n) > 0$ at $\underline{w}_n$, therefore, the only intersection point of Δ_o and Δ_{nc} is f when $w < \underline{w}_n$. Therefore, it must be the case that $\bar{w}_m < w_m$.

3. By the condition in Assumption 1, $\Delta_{nc}(\underline{w}_n) > 0$, so it must be the case that $w_m < \underline{w}_n$.

4. According the results in Proposition 2, $\underline{w}_n < \bar{w}^o < \bar{w}$.

To sum up, we have shown that

$$f \leq \bar{w}_m < w_m < \underline{w}_n < \bar{w}^o < \bar{w}.$$

A.7 Proof of proposition 4

Proof. We start by deriving an expression for the aggregate TFP using the definition in equation (27). To do so, we derive the numerator and denominator respectively. Using the relationship between MPK and capital K in equation (24), we have

$$K_i = \left[\frac{MPK_i}{\alpha(\kappa z)^{1-\alpha}}\right]^{\frac{1}{\alpha-1}}. \tag{A.28}$$

Integrating across all producers in the modern sector to obtain,

$$\int_i K_i di = \int_i \left[\frac{MPK_i}{\alpha(\kappa z)^{1-\alpha}}\right]^{\frac{1}{\alpha-1}} di$$

which further implies that

$$\frac{K_i}{\int_i K_i di} = \frac{\left[\frac{MPK_i}{\alpha(\kappa z)^{1-\alpha}}\right]^{\frac{1}{\alpha-1}}}{\int_i \left[\frac{MPK_i}{\alpha(\kappa z)^{1-\alpha}}\right]^{\frac{1}{\alpha-1}} di}.$$

Therefore, we can calculate the producer-level output Y_i^{mod} as

$$Y_i^{mod} = (\kappa z)^{1-\alpha} \left\{ \frac{\left[\frac{MPK_i}{\alpha(\kappa z)^{1-\alpha}}\right]^{\frac{1}{\alpha-1}}}{\int_i \left[\frac{MPK_i}{\alpha(\kappa z)^{1-\alpha}}\right]^{\frac{1}{\alpha-1}} di} \right\}^{\alpha} \left(\int_i K_i di\right)^{\alpha}$$

Integrating the output across all producers in the modern sector to obtain

$$\begin{aligned} Y &= \int_i Y_i^{mod} di \\ &= \frac{\int_i \left[(\kappa z)^{1-\alpha} \left[\frac{MPK_i}{\alpha(\kappa z)^{1-\alpha}}\right]^{\frac{\alpha}{\alpha-1}} di\right]}{\left(\int_i \left[\frac{MPK_i}{\alpha(\kappa z)^{1-\alpha}}\right]^{\frac{1}{\alpha-1}} di\right)^{\alpha}} \left(\int_i K_i di\right)^{\alpha} \\ &= \frac{\int_i \left[(\kappa z) \left[MPK_i\right]^{\frac{\alpha}{\alpha-1}} di\right]}{\left((\kappa z) \int_i \left[MPK_i\right]^{\frac{1}{\alpha-1}} di\right)^{\alpha}} \left(\int_i K_i di\right)^{\alpha} \end{aligned}$$

Then, according to the definition of TFP in equation (27), we have

$$TFP = \frac{\int_i \left[(\kappa z) \left[MPK_i\right]^{\frac{\alpha}{\alpha-1}} di\right]}{\left((\kappa z) \int_i \left[MPK_i\right]^{\frac{1}{\alpha-1}} di\right)^{\alpha}}. \tag{A.29}$$

Next, we plug in the expression of MPK into equation (A.29) to further derive the TFP for the economy with and without the used capital market.

Specifically, using the expression of MPK in Proposition 2, we calculate the TFP in the economy with the used capital market, $\overline{TFP}$, as:

$$\overline{TFP} = (\kappa z)^{1-\alpha} \frac{\int_{\overline{w}_m}^{\underline{w}_n} \left(\frac{w-f}{q}\right)^{\alpha} di + \int_{\underline{w}_n}^{\overline{w}^o} \left[\frac{q\left(1+\frac{1-\beta q-q}{q(1+\beta\theta)-1}\right)}{\beta\alpha(\kappa z)^{1-\alpha}}\right]^{\frac{\alpha}{\alpha-1}} di + \int_{\overline{w}^o}^{\overline{w}} \left(\frac{w-f}{1-\beta\theta q}\right)^{\alpha} di + \int_{\overline{w}}^{w_{max}} \left(\frac{\bar{w}-f}{1-\beta\theta q}\right)^{\alpha} di}{\left(\int_{\overline{w}_m}^{\underline{w}_n} \frac{w-f}{q} di + \int_{\underline{w}_n}^{\overline{w}^o} \left[\frac{q\left(1+\frac{1-\beta q-q}{q(1+\beta\theta)-1}\right)}{\beta\alpha(\kappa z)^{1-\alpha}}\right]^{\frac{1}{\alpha-1}} di + \int_{\overline{w}^o}^{\overline{w}} \frac{w-f}{1-\beta\theta q} di + \int_{\overline{w}}^{w_{max}} \frac{\bar{w}-f}{1-\beta\theta q} di\right)^{\alpha}} \tag{A.30}$$

where we have used equation (A.18) to calculate the total capital stock for producers with net worth $w \in [\bar{w}, w_{max}]$.

Similarly, the TFP in the economy without used capital, $\widetilde{TFP}$, can be calculated as:

$$\widetilde{TFP} = (\kappa z)^{1-\alpha} \frac{\int_{w_m}^{\underline{w}_n} \left(\frac{w-f}{1-\beta\theta q}\right)^{\alpha} di + \int_{\underline{w}_n}^{\overline{w}^o} \left(\frac{w-f}{1-\beta\theta q}\right)^{\alpha} di + \int_{\overline{w}^o}^{\overline{w}} \left(\frac{w-f}{1-\beta\theta q}\right)^{\alpha} di + \int_{\overline{w}}^{w_{max}} \left(\frac{\bar{w}-f}{1-\beta\theta q}\right)^{\alpha} di}{\left(\int_{w_m}^{\underline{w}_n} \frac{w-f}{1-\beta\theta q} di + \int_{\underline{w}_n}^{\overline{w}^o} \frac{w-f}{1-\beta\theta q} di + \int_{\overline{w}^o}^{\overline{w}} \frac{w-f}{1-\beta\theta q} di + \int_{\overline{w}}^{w_{max}} \frac{\bar{w}-f}{1-\beta\theta q} di\right)^{\alpha}}. \tag{A.31}$$

To compare these two, we define two intermediate statistics. The first one TFP_1 is given by

$$TFP_1 = (\kappa z)^{1-\alpha} \frac{\int_{w_m}^{\underline{w}_n} \left(\frac{w-f}{q}\right)^{\alpha} di + \int_{\underline{w}_n}^{\overline{w}^o} \left[\frac{q\left(1+\frac{1-\beta q-q}{q(1+\beta\theta)-1}\right)}{\beta\alpha(\kappa z)^{1-\alpha}}\right]^{\frac{\alpha}{\alpha-1}} di + \int_{\overline{w}^o}^{\overline{w}} \left(\frac{w-f}{1-\beta\theta q}\right)^{\alpha} di + \int_{\overline{w}}^{w_{max}} \left(\frac{\bar{w}-f}{1-\beta\theta q}\right)^{\alpha} di}{\left(\int_{w_m}^{\underline{w}_n} \frac{w-f}{q} di + \int_{\underline{w}_n}^{\overline{w}^o} \left[\frac{q\left(1+\frac{1-\beta q-q}{q(1+\beta\theta)-1}\right)}{\beta\alpha(\kappa z)^{1-\alpha}}\right]^{\frac{1}{\alpha-1}} di + \int_{\overline{w}^o}^{\overline{w}} \frac{w-f}{1-\beta\theta q} di + \int_{\overline{w}}^{w_{max}} \frac{\bar{w}-f}{1-\beta\theta q} di\right)^{\alpha}},$$

We replace the integration range $[\overline{w}_m, \underline{w}_n]$ in the first integration component (both in the numerator and denominator) with $[w_m, \underline{w}_n]$ in the expression of $\overline{TFP}$ (equation (A.30)) to get TFP_1. The second statistic TFP_2 is given by

$$TFP_2 = (\kappa z)^{1-\alpha} \frac{\int_{w_m}^{\underline{w}_n} \left(\frac{w-f}{q}\right)^{\alpha} di + \int_{\underline{w}_n}^{\overline{w}^o} \left(\frac{w-f}{1-\beta\theta q}\right)^{\alpha} di + \int_{\overline{w}^o}^{\overline{w}} \left(\frac{w-f}{1-\beta\theta q}\right)^{\alpha} di + \int_{\overline{w}}^{w_{max}} \left(\frac{\bar{w}-f}{1-\beta\theta q}\right)^{\alpha} di}{\left(\int_{w_m}^{\underline{w}_n} \frac{w-f}{q} di + \int_{\underline{w}_n}^{\overline{w}^o} \frac{w-f}{1-\beta\theta q} di + \int_{\overline{w}^o}^{\overline{w}} \frac{w-f}{1-\beta\theta q} di + \int_{\overline{w}}^{w_{max}} \frac{\bar{w}-f}{1-\beta\theta q} di\right)^{\alpha}}.$$

To get TFP_2, we replace the term of the first integral (i.e., $\frac{w-f}{1-\beta\theta q}$, both in numerator and denominator) with $\frac{w-f}{q}$ in the expression of $\widetilde{TFP}$ in equation (A.31).

In what follows, we show that

$$\overline{TFP} > TFP_1 > TFP_2 > \widetilde{TFP}. \tag{A.32}$$

We proceed in 3 steps. In step 1, we compare TFP_2 and $\widetilde{TFP}$. To do so, We define

$$F(x) = (\kappa z)^{1-\alpha} \frac{\int_{w_m}^{\underline{w}_n} x^\alpha di + \int_{\underline{w}_n}^{\bar{w}^o} \left(\frac{w-f}{1-\beta\theta q}\right)^\alpha di + \int_{\bar{w}^o}^{\bar{w}} \left(\frac{w-f}{1-\beta\theta q}\right)^\alpha di + \int_{\bar{w}}^{w_{max}} \left(\frac{\bar{w}-f}{1-\beta\theta q}\right)^\alpha di}{\left(\int_{w_m}^{\underline{w}_n} x di + \int_{\underline{w}_n}^{\bar{w}^o} \frac{w-f}{1-\beta\theta q} di + \int_{\bar{w}^o}^{\bar{w}} \frac{w-f}{1-\beta\theta q} di + \int_{\bar{w}}^{w_{max}} \frac{\bar{w}-f}{1-\beta\theta q} di\right)^\alpha}.$$

where $x \in \left[\frac{w-f}{1-\beta\theta q}, \frac{w-f}{q}\right]$, $w \in [w_m, \underline{w}_n]$. Taking a log to obtain,

$$\begin{aligned} f(x) =& log(F(x)) = log\left[(\kappa z)^{1-\alpha}\right] \\ &+ log\left[\int_{w_m}^{\underline{w}_n} x^\alpha di + \int_{\underline{w}_n}^{\bar{w}^o} \left(\frac{w-f}{1-\beta\theta q}\right)^\alpha di + \int_{\bar{w}^o}^{\bar{w}} \left(\frac{w-f}{1-\beta\theta q}\right)^\alpha di + \int_{\bar{w}}^{w_{max}} \left(\frac{\bar{w}-f}{1-\beta\theta q}\right)^\alpha di\right] \\ &- \alpha log\left[\int_{w_m}^{\underline{w}_n} x di + \int_{\underline{w}_n}^{\bar{w}^o} \frac{w-f}{1-\beta\theta q} di + \int_{\bar{w}^o}^{\bar{w}} \frac{w-f}{1-\beta\theta q} di + \int_{\bar{w}}^{w_{max}} \frac{\bar{w}-f}{1-\beta\theta q} di\right]. \end{aligned}$$

To see how $f(x)$ changes with x, we calculate the derivative $\frac{\partial f(x)}{\partial x}$ as

$$\begin{aligned} \frac{\partial f(x)}{\partial x} =& \frac{\int_{w_m}^{\underline{w}_n} \alpha x^{\alpha-1} di}{\int_{w_m}^{\underline{w}_n} x^\alpha di + \int_{\underline{w}_n}^{\bar{w}^o} \left(\frac{w-f}{1-\beta\theta q}\right)^\alpha di + \int_{\bar{w}^o}^{\bar{w}} \left(\frac{w-f}{1-\beta\theta q}\right)^\alpha di + \int_{\bar{w}}^{w_{max}} \left(\frac{\hat{w}-f}{1-\beta\theta q}\right)^\alpha di} \\ &- \alpha \frac{\int_{\underline{w}_n}^{\bar{w}^o} di}{\int_{w_m}^{\underline{w}_n} x di + \int_{\underline{w}_n}^{\bar{w}^o} \frac{w-f}{1-\beta\theta q} di + \int_{\bar{w}^o}^{\bar{w}} \frac{w-f}{1-\beta\theta q} di + \int_{\bar{w}}^{w_{max}} \frac{\hat{w}-f}{1-\beta\theta q} di} \\ =& \alpha \frac{\int_{w_m}^{\underline{w}_n} di}{\int_{w_m}^{\underline{w}_n} x di + \int_{\underline{w}_n}^{\bar{w}^o} \frac{\left(\frac{w-f}{1-\beta\theta q}\right)^\alpha}{x^{\alpha-1}} di + \int_{\bar{w}^o}^{\bar{w}} \frac{\left(\frac{w-f}{1-\beta\theta q}\right)^\alpha}{x^{\alpha-1}} di + \int_{\bar{w}}^{w_{max}} \frac{\left(\frac{\bar{w}-f}{1-\beta\theta q}\right)^\alpha}{x^{\alpha-1}} di} \\ &- \alpha \frac{\int_{\underline{w}_n}^{\bar{w}^o} di}{\int_{w_m}^{\underline{w}_n} x di + \int_{\underline{w}_n}^{\bar{w}^o} \frac{w-f}{1-\beta\theta q} di + \int_{\bar{w}^o}^{\bar{w}} \frac{w-f}{1-\beta\theta q} di + \int_{\bar{w}}^{w_{max}} \frac{\bar{w}-f}{1-\beta\theta q} di}. \end{aligned}$$

We know that $x \in \left[\frac{w-f}{1-\beta\theta q}, \frac{w-f}{q}\right]$, $w \in [w_m, \underline{w}_n]$. Then, when $w > \underline{w}_n$, we have $x \leq$

$\frac{w-f}{1-\beta\theta q} < \frac{\hat{w}-f}{1-\beta\theta q}$. As a result, we have

$$\frac{\left(\frac{w-f}{1-\beta\theta q}\right)^{\alpha-1}}{x^{\alpha-1}} \leq 1 \Rightarrow \frac{\left(\frac{w-f}{1-\beta\theta q}\right)^{\alpha}}{x^{\alpha-1}} = \frac{w-f}{1-\beta\theta q}\frac{\left(\frac{w-f}{1-\beta\theta q}\right)^{\alpha-1}}{x^{\alpha-1}} \leq \frac{w-f}{1-\beta\theta q}.$$

$f(x)$ and $F(x)$ is increasing in x since the first term in $\frac{\partial f(x)}{\partial x}$ has a smaller denominator than the second term. This implies that

$$F\left(\frac{w-f}{q}\right) = TFP_2 > \widetilde{TFP} = F\left(\frac{w-f}{1-\beta\theta q}\right)$$

In step 2, we compare TFP_1 and TFP_2. To do so, we define

$$G(x) = (\kappa z)^{1-\alpha} \frac{\int_{w_m}^{\underline{w}_n} \left(\frac{w-f}{q}\right)^{\alpha} di + \int_{\underline{w}_n}^{\bar{w}^o} x^{\alpha} di + \int_{\bar{w}^o}^{\bar{w}} \left(\frac{w-f}{1-\beta\theta q}\right)^{\alpha} di + \int_{\bar{w}}^{w_{max}} \left(\frac{\bar{w}-f}{1-\beta\theta q}\right)^{\alpha} di}{\left(\int_{w_m}^{\underline{w}_n} \frac{w-f}{q} di + \int_{\underline{w}_n}^{\bar{w}^o} x di + \int_{\bar{w}^o}^{\bar{w}} \frac{w-f}{1-\beta\theta q} di + \int_{\bar{w}}^{w_{max}} \frac{\bar{w}-f}{1-\beta\theta q} di\right)^{\alpha}}$$

where $x \in \left[\frac{w-f}{1-\beta\theta q}, \left(\frac{q\left(1+\frac{1-\beta q-q}{q(1+\beta\theta)-1}\right)}{\beta\alpha(\kappa z)^{1-\alpha}}\right)^{\frac{1}{\alpha-1}}\right]$, and $w \in (\underline{w}_n, \bar{w}^o)$. Again, taking a log to obtain

$$\begin{aligned} g(x) =& log(G(x)) = log\left[(\kappa z)^{1-\alpha}\right] \\ &+ log\left[\int_{w_m}^{\underline{w}_n} \left(\frac{w-f}{q}\right)^{\alpha} di + \int_{\underline{w}_n}^{\bar{w}^o} x^{\alpha} di + \int_{\bar{w}^o}^{\bar{w}} \left(\frac{w-f}{1-\beta\theta q}\right)^{\alpha} di + \int_{\bar{w}}^{w_{max}} \left(\frac{\hat{w}-f}{1-\beta\theta q}\right)^{\alpha} di\right] \\ &- \alpha log\left[\int_{w_m}^{\underline{w}_n} \frac{w-f}{q} di + \int_{\underline{w}_n}^{\bar{w}^o} x di + \int_{\bar{w}^o}^{\bar{w}} \frac{w-f}{1-\beta\theta q} di + \int_{\bar{w}}^{w_{max}} \frac{\hat{w}-f}{1-\beta\theta q} di\right]. \end{aligned}$$

To see how $g(x)$ changes with x, we further take derivative to obtain

$$\begin{aligned}\frac{\partial g(x)}{\partial x}=&\frac{\int_{\underline{w}_n}^{\bar{w}^o}\alpha x^{\alpha-1}di}{\int_{w_m}^{\underline{w}_n}\left(\frac{w-f}{q}\right)^{\alpha}di+\int_{\underline{w}_n}^{\bar{w}^o}x^{\alpha}di+\int_{\bar{w}^o}^{\bar{w}}\left(\frac{w-f}{1-\beta\theta q}\right)^{\alpha}di+\int_{\bar{w}}^{w_{max}}\left(\frac{\hat{w}-f}{1-\beta\theta q}\right)^{\alpha}di}\\
&-\alpha\frac{\int_{\underline{w}_n}^{\bar{w}^o}di}{\int_{w_m}^{\underline{w}_n}\frac{w-f}{q}di+\int_{\underline{w}_n}^{\bar{w}^o}xdi+\int_{\bar{w}^o}^{\bar{w}}\frac{w-f}{1-\beta\theta q}di+\int_{\bar{w}}^{w_{max}}\frac{\bar{w}-f}{1-\beta\theta q}di}\\
=&\alpha\frac{\int_{\underline{w}_n}^{\bar{w}^o}di}{\int_{w_m}^{\underline{w}_n}\frac{\left(\frac{w-f}{q}\right)^{\alpha}}{x^{\alpha-1}}di+\int_{\underline{w}_n}^{\bar{w}^o}xdi+\int_{\bar{w}^o}^{\bar{w}}\frac{\left(\frac{w-f}{1-\beta\theta q}\right)^{\alpha}}{x^{\alpha-1}}di+\int_{\bar{w}}^{w_{max}}\frac{\left(\frac{\bar{w}-f}{1-\beta\theta q}\right)^{\alpha}}{x^{\alpha-1}}di}\\
&-\alpha\frac{\int_{\underline{w}_n}^{\bar{w}^o}di}{\int_{w_m}^{\underline{w}_n}\frac{w-f}{q}di+\int_{\underline{w}_n}^{\bar{w}^o}xdi+\int_{\bar{w}^o}^{\bar{w}}\frac{w-f}{1-\beta\theta q}di+\int_{\bar{w}}^{w_{max}}\frac{\bar{w}-f}{1-\beta\theta q}di}.\end{aligned}$$

Given $x\in\left[\frac{w-f}{1-\beta\theta q},\left(\frac{q\left(1+\frac{1-\beta q-q}{q(1+\beta\theta)-1}\right)}{\beta\alpha(\kappa z)^{1-\alpha}}\right)^{\frac{1}{\alpha-1}}\right]$, and $w\in(\underline{w}_n,\bar{w}^o)$. Then, when $w>\bar{w}^o$, we have $x\leq\frac{w-f}{1-\beta\theta q}$; when $w<\underline{w}_n$, we have $x\leq\frac{w-f}{q}$. Therefore, we obtain the following relationship:

$$\begin{aligned}\frac{\left(\frac{w-f}{1-\beta\theta q}\right)^{\alpha}}{x^{\alpha-1}}&\leq\frac{w-f}{1-\beta\theta q},w>\bar{w}^o\\
\frac{\left(\frac{\hat{w}-f}{1-\beta\theta q}\right)^{\alpha}}{x^{\alpha-1}}&\leq\frac{\hat{w}-f}{1-\beta\theta q},w>\bar{w}^o\\
\frac{\left(\frac{w-f}{q}\right)^{\alpha}}{x^{\alpha-1}}&\leq\frac{w-f}{q},w<\underline{w}_n.\end{aligned}$$

$g(x)$ and $G(x)$ is increasing in x since the first term in $\frac{\partial g(x)}{\partial x}$ has a smaller denominator than the second term. This implies that

$$G\left(\left(\frac{q\left(1+\frac{1-\beta q-q}{q(1+\beta\theta)-1}\right)}{\beta\alpha(\kappa z)^{1-\alpha}}\right)^{\frac{1}{\alpha-1}}\right)=TFP_1>TFP_2=G\left(\frac{w-f}{1-\beta\theta q}\right)$$

In step 3, we compare $\overline{TFP}$ and TFP_1. To do so, we define $H(p)$ as:

$$H(p) = (\kappa z)^{1-\alpha} \frac{\int_{w_m-p}^{\underline{w}_n} \left(\frac{w-f}{q}\right)^{\alpha} di + \int_{\underline{w}_n}^{\bar{w}^o} \left[\frac{q\left(1+\frac{1-\beta q-q}{q(1+\beta\theta)-1}\right)}{\beta\alpha(\kappa z)^{1-\alpha}}\right]^{\frac{\alpha}{\alpha-1}} di + \int_{\bar{w}^o}^{\bar{w}} \left(\frac{w-f}{1-\beta\theta q}\right)^{\alpha} di + \int_{\bar{w}}^{w_{max}} \left(\frac{\bar{w}-f}{1-\beta\theta q}\right)^{\alpha} di}{\left(\int_{w_m-p}^{\underline{w}_n} \frac{w-f}{q} di + \int_{\underline{w}_n}^{\bar{w}^o} \left[\frac{q\left(1+\frac{1-\beta q-q}{q(1+\beta\theta)-1}\right)}{\beta\alpha(\kappa z)^{1-\alpha}}\right]^{\frac{1}{\alpha-1}} di + \int_{\bar{w}^o}^{\bar{w}} \frac{w-f}{1-\beta\theta q} di + \int_{\bar{w}}^{w_{max}} \frac{\bar{w}-f}{1-\beta\theta q} di\right)^{\alpha}}.$$

Taking log, and let $h(p) = \log(H(p))$, we have

$$\begin{aligned}
&log\left[(\kappa z)^{1-\alpha}\right] + \\
&log\left[\int_{w_m-p}^{\underline{w}_n} \left(\frac{w-f}{q}\right)^{\alpha} di + \int_{\underline{w}_n}^{\bar{w}^o} \left[\frac{q\left(1+\frac{1-\beta q-q}{q(1+\beta\theta)-1}\right)}{\beta\alpha(\kappa z)^{1-\alpha}}\right]^{\frac{\alpha}{\alpha-1}} di + \int_{\bar{w}^o}^{\bar{w}} \left(\frac{w-f}{1-\beta\theta q}\right)^{\alpha} di + \int_{\bar{w}}^{w_{max}} \left(\frac{\bar{w}-f}{1-\beta\theta q}\right)^{\alpha} di\right] \\
&-\alpha log\left[\int_{w_m-p}^{\underline{w}_n} \frac{w-f}{q} di + \int_{\underline{w}_n}^{\bar{w}^o} \left[\frac{q\left(1+\frac{1-\beta q-q}{q(1+\beta\theta)-1}\right)}{\beta\alpha(\kappa z)^{1-\alpha}}\right]^{\frac{1}{\alpha-1}} di + \int_{\bar{w}^o}^{\bar{w}} \frac{w-f}{1-\beta\theta q} di + \int_{\bar{w}}^{w_{max}} \frac{\bar{w}-f}{1-\beta\theta q} di\right].
\end{aligned}$$

To see how $h(p)$ changes with p, we take derivative to obtain

$$\begin{aligned}
\frac{\partial h(p)}{\partial p} =& \frac{\left(\frac{w_m-p-f}{q}\right)^{\alpha}}{\int_{w_m-p}^{\underline{w}_n} \left(\frac{w-f}{q}\right)^{\alpha} di + \int_{\underline{w}_n}^{\bar{w}^o} \left[\frac{q\left(1+\frac{1-\beta q-q}{q(1+\beta\theta)-1}\right)}{\beta\alpha(\kappa z)^{1-\alpha}}\right]^{\frac{\alpha}{\alpha-1}} di + \int_{\bar{w}^o}^{\bar{w}} \left(\frac{w-f}{1-\beta\theta q}\right)^{\alpha} di + \int_{\bar{w}}^{w_{max}} \left(\frac{\bar{w}-f}{1-\beta\theta q}\right)^{\alpha} di} \\
&-\alpha \frac{\frac{w_m-p-f}{q}}{\int_{w_m-p}^{\underline{w}_n} \frac{w-f}{q} di + \int_{\underline{w}_n}^{\bar{w}^o} \left[\frac{q\left(1+\frac{1-\beta q-q}{q(1+\beta\theta)-1}\right)}{\beta\alpha(\kappa z)^{1-\alpha}}\right]^{\frac{1}{\alpha-1}} di + \int_{\bar{w}^o}^{\bar{w}} \frac{w-f}{1-\beta\theta q} di + \int_{\bar{w}}^{w_{max}} \frac{\bar{w}-f}{1-\beta\theta q} di} \\
=& \frac{1}{\int_{w_m-p}^{\underline{w}_n} \left(\frac{\frac{w-f}{q}}{\frac{w_m-p-f}{q}}\right)^{\alpha} di + \int_{\underline{w}_n}^{\bar{w}^o} \left[\frac{\left(\frac{q\left(1+\frac{1-\beta q-q}{q(1+\beta\theta)-1}\right)}{\beta\alpha(\kappa z)^{1-\alpha}}\right)^{\frac{1}{\alpha-1}}}{\frac{w_m-p-f}{q}}\right]^{\alpha} di + \int_{\bar{w}^o}^{\bar{w}} \left(\frac{\frac{w-f}{1-\beta\theta q}}{\frac{w_m-p-f}{q}}\right)^{\alpha} di + \int_{\bar{w}}^{w_{max}} \left(\frac{\frac{\bar{w}-f}{1-\beta\theta q}}{\frac{w_m-p-f}{q}}\right)^{\alpha} di} \\
&-\alpha \frac{1}{\int_{w_m-p}^{\underline{w}_n} \frac{\frac{w-f}{q}}{\frac{w_m-p-f}{q}} di + \int_{\underline{w}_n}^{\bar{w}^o} \frac{\left[\frac{q\left(1+\frac{1-\beta q-q}{q(1+\beta\theta)-1}\right)}{\beta\alpha(\kappa z)^{1-\alpha}}\right]^{\frac{1}{\alpha-1}}}{\frac{w_m-p-f}{q}} di + \int_{\bar{w}^o}^{\bar{w}} \frac{\frac{w-f}{1-\beta\theta q}}{\frac{w_m-p-f}{q}} di + \int_{\bar{w}}^{w_{max}} \frac{\frac{\bar{w}-f}{1-\beta\theta q}}{\frac{w_m-p-f}{q}} di}
\end{aligned}$$

We have the following relationships:

- If $w \in [w_m - p, \underline{w}_n)$

$$\frac{\frac{w-f}{q}}{\frac{w_m-p-f}{q}} > 1$$

- If $w \in [\underline{w}_n, \bar{w}^o)$

$$\frac{\left[\frac{q\left(1+\frac{1-\beta q-q}{q(1+\beta\theta)-1}\right)}{\beta\alpha(\kappa z)^{1-\alpha}}\right]^{\frac{1}{\alpha-1}}}{\frac{w_m-p-f}{q}} > 1$$

- If $w \in [\bar{w}^o, \bar{w})$

$$\frac{\frac{w-f}{1-\beta\theta q}}{\frac{w_m-p-f}{q}} > 1$$

- If $w \geq \bar{w}$

$$\frac{\frac{\bar{w}-f}{1-\beta\theta q}}{\frac{w_m-p-f}{q}} > 1$$

Since $0 < \alpha < 1$, then it follows that

$$\frac{\partial h(p)}{\partial p} > 0$$

Thus, $\overline{TFP} > TFP_1$, and we obtain

$$\overline{TFP} > TFP_1 > TFP_2 > \widetilde{TFP}.$$

□

A.8 Proof of proposition 5

Proof. We first derive the efficient TFP TFP^e. We start from the definition of total TFP derived in equation (A.29),

$$TFP = \frac{\int_i \left[(\kappa z)\left[MPK_i\right]^{\frac{\alpha}{\alpha-1}} di\right]}{\left((\kappa z)\int_i \left[MPK_i\right]^{\frac{1}{\alpha-1}} di\right)^{\alpha}}.$$

At the efficient allocation, all $MPKs$ are equalized, so we have:

$$TFP^e = \frac{\int_i [(\kappa z)di]}{\left((\kappa z)\int_i di\right)^\alpha} = \left[\int_i (\kappa z)di\right]^{1-\alpha}.$$

Given we have proven that $\overline{w}_m < w_m$ in Proposition 3, meaning that there are more producers in the modern sector when we have used capital market, so we can conclude that

$$\overline{TFP^e} > \widetilde{TFP^e}. \tag{A.33}$$

This implies that the efficient TFP in the economy with a used capital market is larger than that in the economy without a used capital market. According to the definition of G^{TFP}_{entry} in equation (35), it follows that

$$G^{TFP}_{entry} > 0.$$

Moreover, producers with a net worth larger than w_m, will enter the modern section even when there is a used capital market. Since used capital is easier to finance, it allows more constrained producers to employ more used capital, generating a more even MPK across producers. Therefore, it is immediate that

$$G^{TFP}_{misall} > 0.$$

□

B Appendix to the quantitative model

B.1 Recursive formulation

As in Midrigan and Xu (2014), we solve the model using the value function iteration method. To do so, we first rewrite the producers' optimization problem of each type of producer in recursive form.

Households (Workers): The households' problem is standard and can be rewritten as

$$V\left(B^h_{t-1}, v_t\right) = \log\left(C^w_t\right) + \beta E_t V\left(D^h_t, v_{t+1}\right) \tag{B.1}$$

subject to the budget constraint

$$C_t^w + B_t^h = W_t\left(\iota\right)\gamma^t v_t + (1+r)B_{t-1}^h \tag{B.2}$$

Producers in modern sector: We define a producer's net worth at time $t+1$ by $N_t = qK_t^o + K_t^n - B_t^{mod}$. Since profits, output, and the optimal choice of capital and labor are homogeneous of degree one in net worth N and permanent productivity $\exp(z)$, so we re-scale all variables by $exp(z)$. We use the lowercase $x = \frac{X}{exp(z)}$ to denote the normalized variables. The Bellman equation along a balanced growth path with constant prices W, r, and q of a producer with rescaled net worth $n_{t-1} = N_{t-1}/exp(z)$ and productivity e_t is given by

$$V^{mod}\left(n_{t-1}, e_t\right) = \max_{n_t, c_t^{mod}} \log\left(c_t^{mod}\right) + \beta E_t V^{mod}\left(n_t, e_{t+1}\right), \tag{B.3}$$

subject to the budget constraint

$$c^{mod} + n_t = \pi_t^{mod}\left(e_t\right) + (1+r)n_{t-1}, \tag{B.4}$$

where

$$\pi_t^{mod}(e_t) = \max_{k_{t-1}^n, k_{t-1}^o, l_t} \exp\left(e_t + \kappa\right)^{1-\eta}\left(l_t^{1-\alpha}k_{t-1}^{\alpha}\right)^{\eta} - \left(r + \delta^n\left(1-q\right)\right)k_{t-1}^n - q\left(r+\delta^o\right)k_{t-1}^o - Wl_t. \tag{B.5}$$

The borrowing constraint was reduced to

$$(1+r)\left(qk_{t-1}^o + k_{t-1}^n - n_{t-1}\right) \leq \theta\left[\left(1-\delta^n\left(1-q\right)\right)k_{t-1}^n + q\left(1-\delta^o\right)k_{t-1}^o\right] \tag{B.6}$$

Equations (B.4)-(B.6) simply rewrite the budget constraint in Equation (42) and the borrowing constraint in Equation (40) of producers in the modern sector using the new notations.

Producers in traditional sector: Let $N_t = -B_t^{tra}$ denote the time $t+1$ net worth of a producer in the traditional sector. With rescaled net worth $n_t = N_t/exp(z)$ and productivity e_t, the Bellman equation of such producers is given by

$$V^{tra}\left(n_{t-1}, e_t\right) = \max_{n_t, c_t^{tra}} \log\left(c_t^{tra}\right) + \beta \max\left\{E_t V^{tra}\left(n_t, e_{t+1}\right), E_t V^{mod}\left(n_t, e_{t+1}\right)\right\} \tag{B.7}$$

subject to the budget constraint

$$c_t^{tra} = \pi_t^{tra}(e_t) + (1+r)n_{t-1} - \left\{n\right\}\mathbb{1}_{tra} + \left\{-n_t - exp(z)\kappa\right\}\mathbb{1}_{mod} \tag{B.8}$$

where

$$\pi_t^{tra}(e_t) = \max_{l_t} \exp(e_t)^{1-\eta}(l_t)^{\eta} - Wl_t. \tag{B.9}$$

are the profits a producer in the traditional sector, and $n_t = -b_t^{tra}$ are its savings.

The producer's continuation value is the envelope over the expected value of the two options it has: staying in the traditional sector or switching to the modern sector. The evolution of its net worth is a function of whether the producer switches. A producer that stays in the traditional sector simply inherits its past savings. In contrast, a producer that enters the modern sector has to pay a fix cost f.

B.2 Law of motion of producer measures

We denote the measure of the traditional sector by $\Phi_t^{tra}(n,e)$, and the measure of the modern sector by $\Phi_t^{mod}(n,e)$. The measures of producers in the two sectors must add up to $M_t = \gamma^t$:

$$\int_{\Lambda\times E} d\Phi_t^{tra}(n,e) + \int_{\Lambda\times E} d\Phi_t^{mod}(n,e) = \gamma^t.$$

Furthermore, to characterize the evolution of these measures, let $\zeta(n,e)$ be an indicator for whether a producer in the traditional sector switches to the modern sector. The measure of producers in the traditional sector $\Phi_t^{tra}(n,e)$ evolves over time according to

$$\begin{aligned}\Phi_{t+1}^{tra}(n',e_j) = &\int_{\Lambda}\sum_i p_{i,j} I_{\{\zeta(n,e_i)=0,n^{tra}(n,e_i)\in A\}} d\Phi_t^{tra}(n,e_i)\\ &+ (\gamma-1)\gamma^t I_{\{0\in\Lambda\}}\bar{p}_j,\end{aligned} \tag{B.10}$$

where $n^{tra}(n,e_i)$ is the savings decision of a producer that remains in the traditional sector, $\bar{p}_j$ is the stationary distribution of the transitional productivity, n' denotes time $t+1$, idiosyncratic productivity jump from e_i at time t to e_j at time $t+1$. The measure adds up producers that stay in the traditional sector and newly entering producers.

Moreover, the measure of producers in the modern sector $\Phi_t^{mod}(n,e)$ contains producers already in the modern sector and producers who decided to enter the modern sector. So, it evolves according to

$$\begin{aligned}\Phi_{t+1}^{mod}(n',e_j) = &\int_{\Lambda}\sum_i p_{i,j} I_{\{n^{mod}(n,e_i)\in\Lambda\}} d\Phi_t^{mod}(n,e_i)\\ &+ \int_{\Lambda}\sum_i p_{i,j} I_{\{\zeta(n,e_i)=1,n^{tramod}(n,e_i)\in\Lambda\}} d\Phi_t^{tra}(n,e_i),\end{aligned} \tag{B.11}$$

where $n^{mod}(.)$ is the decision rule for producers in the modern sector. $n^{tramod}(.)$ is the amount of net worth a producer that switches sectors carries into the next period.

B.3 Market clearing conditions

We have the following market-clearing conditions:

1. The labor market clearing condition:

$$\int_{\Lambda\times E} l^{tra}(e)d\Phi_t^{tra}(n,e) + \int_{\Lambda\times E} l^{mod}(n,e)d\Phi_t^{mod}(n,e) = L_t,$$

   where $L_t = \gamma^t$ is the efficient units of labor supplied by household.

2. The used capital market clearing condition:

$$\int_{\Lambda\times E} k^o(n,e)d\Phi_t^{mod}(n,e) = \int_{\Lambda\times E}(1-\delta_o)k^o(n,e)d\Phi_t^{mod}(n,e) + \int_{\Lambda\times E} \delta_n k^n(n,e)d\Phi_t^{mod}(n,e),$$

3. The goods market clearing condition:

$$\begin{aligned}\int_{\Lambda\times V} c^h\,(n,v)\,dv &+ \int_{\Lambda\times E} c^{mod}(n,e)d\Phi_t^{mod}(n,e) + \int_{\Lambda\times E} c^{tra}(n,e)I_{\{\zeta(n,e)=0,n^{tra}(n,e)\in\Lambda\}}d\Phi_t^{tra}(n,e)\\ &+ \int_{\Lambda\times E}\left(c^{tramod}(n,e)+f\right)I_{\{\zeta(n,e)=1,n^{tramod}(n,e)\in\Lambda\}}d\Phi_t^{tra}(n,e) + \int_{\Lambda\times E}\delta_n k^n(n,e)d\Phi_t^{mod}(n,e)\\ &= \int_{\Lambda\times E} y^{tra}(e)d\Phi_t^{tra}(n,e) + \int_{\Lambda\times E} y^{mod}(n,e)d\Phi_t^{mod}(n,e).\end{aligned}$$

B.4 Solution Methods

In this subsection we will briefly introduce the related useful FOCs and solution processes to the quantitative models that underlie the quantitative analyses. The households' problem is solved via endogenous grid method and producers' problems are solved via value function iteration.

B.4.1 Households' Problem

$$V\left(B_{t-1}^h, v_t\right) = \log\left(C_t^w\right) + \beta E_t V\left(B_t^h, v_{t+1}\right) \tag{B.12}$$

$$C_t^w + B_t^h = W\gamma^t v_t + (1+r)B_{t-1}^h \tag{B.13}$$

To ensure the stationarity of the contraction mapping, we detrend the households' budget constraint by the productivity growth rate γ. Meanwhile, to further simplify the solution steps, we assume an extra permanent labor productivity ι, in addition to the idiosyncratic labor productivity v_t, to help to normalize the wage at 1 under stationary equilibrium. Hence, we can rewrite the budget constraint (B.13) into

$$c_t^w + b_t^h = Wv_t + \frac{1+r_t}{\gamma} b_{t-1}^h \tag{B.14}$$

where $c_t^w = \frac{C_t^w}{\gamma^t \iota}$ and $b_t^h = \frac{B_t^h}{\gamma^t \iota}$.

The FOCs and envelop condition will be

$$\beta E_t V_{B^h,t} = \lambda_t^w$$

$$\frac{1}{c_t^w} = \lambda_t^w$$

$$V_{B^h,t-1} = \lambda_t^w \frac{1+r_t}{\gamma}$$

Thus the Euler equation will be

$$\frac{1}{c_t^w} = \beta \frac{1+r_t}{\gamma} E_t \frac{1}{c_{t+1}^w} \tag{B.15}$$

Based on this Euler equation we can solve the Households' problem by means of the standard endogenous grid method, which we show in detail below.

Set the grid on b_t^h and guess the function $b_{t+1}^h\left(b_t^h, v_{t+1}\right) = b_{t+1}^{h,0}$

- Step 1: Calculate the RHS of Euler equation (B.15) $RHS = \frac{1+r_t}{\gamma}\beta E_t \frac{1}{Wv_{t+1} + \frac{1+r_t}{\gamma} b_t^h - b_{t+1}^{h,0}}$
- Step 2: Calculate b_t^h through the Euler equation and budget. constraint $b_{t-1}^h = \frac{\gamma}{1+r}\left(\frac{1}{RHS} + b_t^h - Wv_t\right)$.
- Step 3: Update the guess $b_t^h\left(b_{t-1}^h, v_t\right)$ froward to $b_{t+1}^h\left(b_t^h, v_{t+1}\right) = b_{t+1}^{h,1}$ and go back to step 1. Stop until $\left\|b_{t+1}^{h,n-1} - b_{t+1}^{h,n}\right\| < \varepsilon$.

B.4.2 Modern Sector

For simplicity, rewrite the sub-period problem as[B.1]

$$\pi_t^m(e) = \max_{k_{t-1}^n, k_{t-1}^o, l_t} a_t^{1-\eta}\left(l_t^{1-\alpha}k_{t-1}^{\alpha}\right)^{\eta} - \left(r+\delta^n(1-q)\right)k_{n,t-1} - q\left(r+\delta^o\right)k_{o,t-1} - Wl_t \quad \text{(B.16)}$$

where $a_t = \exp\left(e_t + \kappa\right)$.

Non-binding firms will only use the new capital by solving the sub-period problem to get

$$k_{nb,t-1} = k_{n,nb,t-1}$$

$$k_{o,nb,t-1} = 0$$

$$k_{n,nb,t-1} = \left[\varphi_{nb}\left(\frac{W}{(1-\alpha)\eta}\right)^{\frac{(1-\alpha)\eta}{1-(1-\alpha)\eta}} a_t^{\eta-1+\frac{(1-\eta)(1-\alpha)\eta}{(1-\alpha)\eta-1}}\right]^{\eta_\varphi} \quad \text{(B.17)}$$

$$l_{nb,t} = \left[\frac{W}{(1-\alpha)\eta a_t^{1-\eta}k_{n,nb,t-1}^{\alpha\eta}}\right]^{\frac{1}{(1-\alpha)\eta-1}}$$

For binding firms that $k_{o,b,t-1} > 0$ and $k_{n,b,t-1} > 0$

$$k_{b,t-1} = \left[\frac{\varphi_2}{1-\varphi}\left(\frac{W}{(1-\alpha)\eta}\right)^{\frac{(1-\alpha)\eta}{1-(1-\alpha)\eta}} a_t^{\eta-1+\frac{(1-\eta)(1-\alpha)\eta}{(1-\alpha)\eta-1}}\right]^{\eta_\varphi} \quad \text{(B.18)}$$

$$k_{o,b,t-1} = \frac{1}{1-\varphi}\left[k_{b,t-1} - \varphi_3 n_{b,t-1}\right] \quad \text{(B.19)}$$

$$k_{n,b,t-1} = \varphi_3 n_{b,t-1} - \varphi k_{o,b,t-1} \quad \text{(B.20)}$$

$$l_{nb,t} = \left[\frac{W}{(1-\alpha)\eta a_t^{1-\eta}k_{b,t-1}^{\alpha\eta}}\right]^{\frac{1}{(1-\alpha)\eta-1}}$$

where $\varphi_1 = \frac{1+r-\theta(1-\delta^n(1-q))}{(1+r)(1-q)+\theta q(1-\delta^o)-\theta(1-\delta^n(1-q))}$, $\varphi_2 = \frac{q(r+\delta^o)}{\alpha\eta} - \frac{[\theta q(1-\delta^o)-(1+r)q](r+\delta^n(1-q))}{[\theta(1-\delta^n(1-q))-(1+r)]\eta\alpha}$, $\varphi_3 = \frac{1+r}{1+r-\theta(1-\delta^n(1-q))}$, $\varphi = \frac{(1+r)q-\theta q(1-\delta^o)}{1+r-\theta(1-\delta^n(1-q))}$, $\varphi_{nb} = \frac{r+\delta^n(1-q_t)}{\alpha\eta}$, $\eta_\varphi = \frac{1}{\frac{\alpha\eta^2(1-\alpha)}{1-(1-\alpha)\eta}+\alpha\eta-1}$

[B.1] I move the new and old captial's notation from superscription to subscription to avoid mass on exponents.

For binding firms that $k_{o,b,t-1} > 0$ and $k_{n,b,t-1} = 0$

$$k_{o,b,t-1} = \frac{1+r}{q\left[1+r-\theta\left(1-\delta^{o}\right)\right]} n_{b,t-1} \tag{B.21}$$
$$= \frac{\varphi_3}{\varphi} n_{b,t-1}$$

For binding firms that $k_{o,b,t-1} = 0$ and $k_{n,b,t-1} > 0$

$$k_{n,b,t-1} = \frac{1+r}{1+r-\theta\left[1-\delta^{n}\left(1-q\right)\right]} n_{b,t-1}$$
$$= \varphi_3 n_{b,t-1} \tag{B.22}$$

Domain discussion

If the same results hold as in the two-period model, we will have $\underline{n}_{b,t-1} < \overline{n}_{b,t-1} < \overline{n}^{n}_{b,t-1}$. Based on guess and verify we can show that this result will also hold in this general quantitative model. Firstly let us guess that the three thresholds exist and their properties follows

1. when $n_{b,t-1} \in [0, \underline{n}_{b,t-1}]$, the constrained entrepreneurs in modern sector will only select the old capital
2. when $n_{b,t-1} \in [\underline{n}_{b,t-1}, \overline{n}_{b,t-1}]$, the constrained entrepreneurs in modern sector will select both the old capital and new capital
3. when $n_{b,t-1} \in \left[\overline{n}_{b,t-1}, \overline{n}^{n}_{b,t-1}\right]$, the constrained entrepreneurs in modern sector will only select the new capital
4. when $n_{b,t-1} > \overline{n}^{n}_{b,t-1}$, the entrepreneurs in modern sector will not be constrained anymore.

Additionally, by observing the equations (B.18-B.20), we can find that the total capital is fixed at $k_{b,t-1}$, when $n_{b,t-1}$ increases, $k_{o,b,t-1}$ decreases and $k_{n,b,t-1}$ increases.

We can solve the $\underline{n}_{b,t-1}$ by setting equation (B.19) equal to the equation (B.21):

$$\frac{1}{1-\varphi}\left[\left[\frac{\varphi_2}{1-\varphi}\left(\frac{W}{(1-\alpha)\eta}\right)^{\frac{(1-\alpha)\eta}{1-(1-\alpha)\eta}} a_t^{\eta-1+\frac{(1-\eta)(1-\alpha)\eta}{(1-\alpha)\eta-1}}\right]^{\eta\varphi} - \varphi_3 n_{b,t-1}\right] = \frac{\varphi_3}{\varphi} n_{b,t-1} \tag{B.23}$$

Alternatively, we can set equation (B.20) equal to zero or equation (B.18) equal to equation (B.21), which is obviously true as long as $k_{b,t-1} = k_{o,b,t-1} + k_{n,b,t-1}$ holds.

We can solve the $\overline{k}^{n}_{b,t-1}$ by setting the equation (B.17) equal to the equation (B.22)

$$k_{n,nb,t-1} = \left[\varphi_{nb}\left(\frac{W_t}{(1-\alpha)\,\eta}\right)^{\frac{(1-\alpha)\eta}{1-(1-\alpha)\eta}} a_t^{\eta-1+\frac{(1-\eta)(1-\alpha)\eta}{(1-\alpha)\eta-1}}\right]^{\eta_\varphi} = \varphi_3 n_{b,t-1} \tag{B.24}$$

Since the equation (B.18) is solved from the scenario that the entrepreneurs select $k_{o,b,t-1}$ and $k_{n,b,t-1}$ freely, under the bounded collateral constraint. Whereas, $\overline{n}^{n}_{b,t-1}$ is the threshold that the firms only select the new capital. Therefore we should not use the equation (B.18) and set $k_{o,b,t-1} = \frac{1}{1-\varphi}\left[k_{b,t-1} - \varphi_3 n_{b,t-1}\right] = 0$, or $k_{b,t-1} = \varphi_3 n_{b,t-1}$.

Furthermore, we can solve the $\overline{n}_{b,t-1}$ descendantly, by setting equation (B.19) equal to zero

$$\left[\frac{\varphi_2}{1-\varphi}\left(\frac{W}{(1-\alpha)\,\eta}\right)^{\frac{(1-\alpha)\eta}{1-(1-\alpha)\eta}} a_t^{\eta-1+\frac{(1-\eta)(1-\alpha)\eta}{(1-\alpha)\eta-1}}\right]^{\eta_\varphi} = \varphi_3 n_{b,t-1} \tag{B.25}$$

Meanwhile if we solve the $\overline{n}_{b,t-1}$ focusing on the $k_{n,b,t-1}$, we should also have the equality between equation (B.20) and equation (B.22), when $k_{o,b,t-1} = 0$, which is obviously true.

By solving equation (B.23):

$$\underline{n}_{b,t-1} = \frac{\varphi}{\varphi_3}\left[\frac{\varphi_2}{1-\varphi}\left(\frac{W}{(1-\alpha)\,\eta}\right)^{\frac{(1-\alpha)\eta}{1-(1-\alpha)\eta}} a_t^{\eta-1+\frac{(1-\eta)(1-\alpha)\eta}{(1-\alpha)\eta-1}}\right]^{\eta_\varphi}$$

By solving equation (B.24):

$$\overline{n}^{n}_{b,t-1} = \frac{1}{\varphi_3}\left[\varphi_{nb}\left(\frac{W}{(1-\alpha)\,\eta}\right)^{\frac{(1-\alpha)\eta}{1-(1-\alpha)\eta}} A_t^{\eta-1+\frac{(1-\eta)(1-\alpha)\eta}{(1-\alpha)\eta-1}}\right]^{\eta_\varphi}$$

By solving equation (B.25):

$$\overline{n}_{b,t-1} = \frac{1}{\varphi_3}\left[\frac{\varphi_2}{1-\varphi}\left(\frac{W}{(1-\alpha)\,\eta}\right)^{\frac{(1-\alpha)\eta}{1-(1-\alpha)\eta}} a_t^{\eta-1+\frac{(1-\eta)(1-\alpha)\eta}{(1-\alpha)\eta-1}}\right]^{\eta_\varphi}$$

Additionally, it is easy to show that as long as $\varphi < 1$, $\underline{n}_{b,t-1} < \overline{n}_{b,t-1}$ will hold, and as long as $\varphi_{nb} < \frac{\varphi_2}{1-\varphi}$ $(\eta_\varphi < 0)$, $\overline{n}_{b,t-1} < \overline{n}^{n}_{b,t-1}$ will hold.

Rearrange $\varphi < 1$ to yield $q < \frac{1+r-\theta(1-\delta^n)}{1+r-\theta(1-\delta^o+\delta^n)}$ and rewrite the inequality $\varphi_{nb} < \frac{\varphi_2}{1-\varphi}$ to

$$q\left(r+\delta^{o}\right)>r+\delta^{n}\left(1-q_{t}\right)$$

These two conditions provide the upper and lower boundary of the used capital price q that satisfies $\underline{n}_{b,t-1}<\overline{n}_{b,t-1}<\overline{n}_{b,t-1}^{n}$

$$\frac{r+\delta^{n}}{r+\delta^{n}+\delta^{o}}<q<\frac{1+r-\theta\left(1-\delta^{n}\right)}{1+r-\theta\left(1-\delta^{o}+\delta^{n}\right)}$$

Given our conjecture, we successfully solve out the three thresholds $\underline{n}_{b,t-1}$, $\overline{n}_{b,t-1}$ and $\overline{n}_{b,t-1}^{n}$. Now let us verify that the within the four intervals entrepreneurs will select within different capital ranges that are analogous.

1. when $n_{b,t-1}\in\left[0,\underline{n}_{b,t-1}\right]$, $k_{o,b,t-1}>0$ and $k_{n,b,t-1}=0$ hold. It is easy to check that $k_{o,b,t-1}$ is monotonic increasing in $n_{b,t-1}$ from equation (B.21).
2. when $n_{b,t-1}\in\left[\underline{n}_{b,t-1},\overline{n}_{b,t-1}\right]$, $k_{o,b,t-1}>0$, $k_{n,b,t-1}>0$ and $k_{o,b,t-1}+k_{n,b,t-1}=\overline{k}_{b,t-1}$ hold.It is easy to check that $k_{o,b,t-1}$ is monotonic decreasing in $n_{b,t-1}$ from equation (B.19) and $k_{n,b,t-1}$ is monotonic increasing in $n_{b,t-1}$ from equation (B.20).
3. when $n_{b,t-1}\in\left[\overline{n}_{b,t-1},\overline{n}_{b,t-1}^{n}\right]$, $k_{o,b,t-1}=0$ and $k_{n,b,t-1}>0$ hold. It is easy to check that $k_{n,b,t-1}$ is monotonic increasing in $n_{b,t-1}$ from equation (B.22).

Based on this value function B.3 we can solve the entrepreneurs problem in modern sector by means of the standard value function iteration, which we show in detail below.

Set the grid on n_t and guess the function $V^{\text{mod}}\left(n_t,e_{t+1}\right)=V_t^0$

- Step 1: Calculate the RHS of value function B.3 through the golden search $RHS=\log\left(\pi_t^m(e_t)+\left(1+r\right)n_{t-1}-n_t\right)+\beta E_t V_t^0$
- Step 2: Calculate V_{t-1}^0 through the value function $V_{t-1}^0=RHS$
- Step 3: Update the guess $V^{\text{mod}}\left(n_{t-1},e_t\right)$ froward to $V^{\text{mod}}\left(n_t,e_{t+1}\right)=V_{t+1}^1$ with a *damping rate* and go back to step 1. Stop until $\left\|V_t^{n-1}-V_{t+1}^n\right\|<\varepsilon$.

B.4.3 Traditional Sector

It is worth to notice that the discrete value function B.7 is different with

$$V^{tra}\left(n_{t-1},e_t\right)=\max\left\{\max_{n_t,c_t^{tra}}\log\left(c_t^{tra}\right)+E_tV^{tra}\left(n_t,e_{t+1}\right),\max_{n_t,c_t^{tra}}\log\left(c_t^{tra}\right)+E_tV^{mod}\left(n_t,e_{t+1}\right)\right\}$$

to ensure the existence of a fix point.

The solution method is also value function iteration, similar to that of the modern sector.

- Step 1: Take an initial guess on $V^{\text{tra}}\left(n_t^{\text{tra}},e_{t+1}\right)$ and solve the cutoff $\widetilde{n}_t$ at which $E_tV^{tra}\left(\widetilde{n}_t,e_{t+1}\right)=E_tV^{mod}\left(\widetilde{n}_t,e_{t+1}\right)$ holds

$$V_{non-jump}^{\text{tra}}\left(N_{t-1}^{\text{tra}},e_t\right)=\underset{N_t,C_t^{tra}}{max}\log\left(C_t^{tra}\right)+E_t\beta V^{\text{tra}}\left(N_t^{\text{tra}},e_{t+1}\right)$$

  and

$$V_{jump}^{\text{tra}}\left(N_{t-1}^{\text{tra}},e_t\right)=\underset{N_t,C_t^{tra}}{max}\log\left(C_t^{tra}\right)+E_t\beta V^{\text{mod}}\left(N_t^{\text{mod}},e_{t+1}\right)$$

  where $V^{\text{mod}}\left(N_t^{\text{mod}},e_{t+1}\right)$ is solved from the modern sector problem.

- Step 2: Solve the problem

$$V_{non-jump}^{\text{tra}}\left(n_{t-1}^{\text{tra}},e_t\right)=\underset{n_t,c_t^{tra}}{max}\log\left(c_t^{tra}\right)+E_t\beta V^{\text{tra}}\left(n_t^{\text{tra}},e_{t+1}\right)$$

  yet subject to the extra constraint $n_t<\widetilde{n}_t$. Similarly, solve the problem

$$V_{jump}^{\text{tra}}\left(n_{t-1}^{\text{tra}},e_t\right)=\underset{n_t,c_t^{tra}}{max}\log\left(c_t^{tra}\right)+E_t\beta V^{\text{mod}}\left(n_t^{\text{mod}},e_{t+1}\right)$$

  and subject to the extra constraint $n_t\geq\widetilde{n}_t$.

- Step 3: Comparing $V_{non-jump}^{\text{tra}}\left(n_{t-1}^{\text{tra}},e_t\right)$ and $V_{jump}^{\text{tra}}\left(n_{t-1}^{\text{tra}},e_t\right)$ to yield $V^{\text{tra}}\left(n_{t-1}^{\text{tra}},e_t\right)=\max\left\{\log\left(c_{t,non-jump}^{tra}\right)+V_{non-jump}^{\text{tra}}\left(n_{t,non-jump}^{\text{tra}},e_t\right),\log\left(c_{t,jump}^{tra}\right)+V_{jump}^{\text{tra}}\left(n_{t,jump}^{\text{tra}},e_t\right)\right\}$ where $n_{t,non-jump}^{\text{tra}}=\text{argmax}\log\left(c_t^{tra}\right)+E_t\beta V^{\text{tra}}\left(n_t^{\text{tra}},e_{t+1}\right)$ and $n_{t,jump}^{\text{tra}}=\text{argmax}\log\left(c_t^{tra}\right)+E_t\beta V^{\text{mod}}\left(n_t^{\text{mod}},e_{t+1}\right)$, which we solved at step 2.

- Step 3: Update the $V^{\text{tra}}\left(n_{t-1}^{\text{tra}},e_t\right)$ until it becomes stable.

B.4.4 Aggregation and market clearing

Since we solve either the households' problem B.12 or the entrepreneurs' problems with stationary transformation, to clear the market we should also adjust the original marketing clearing conditions to attain the balance growth path.

The original aggregation of the labor market is unstable as

$$\int_{\Lambda\times E} L^{\text{tra}}(n,e)d\Phi_t^{\text{tra}}(n,e) + \int_{\Lambda\times E} L^{\text{mod}}(n,e)d\Phi_t^{\text{mod}}(n,e) = \gamma^t \tag{B.26}$$

Therefore we divided both side with γ^t and define the effective measurement of the mass as $\Phi^{\text{tra}}(n,e) = \frac{\Phi_t^{\text{tra}}(n,e)}{\gamma^t}$ and $\Phi^{\text{mod}}(n,e) = \frac{\Phi_t^{\text{mod}}(n,e)}{\gamma^t}$ to yield

$$\int_{\Lambda\times E} d\Phi_{n,e}^{\text{tra}} + \int_{\Lambda\times E} d\Phi_{n,e}^{\text{mod}} = \iota \tag{B.27}$$

The bonds market clearing condition becomes

$$\int_{\Lambda\times E} B_t^{\text{tra}}(e)d\Phi_t^{\text{tra}}(a,e) + \int_{\Lambda\times E} B_t^{\text{mod}}(a,e)d\Phi_t^{\text{mod}}(a,e) = \int_V B_t^h(\nu)d\Phi_t^{\text{household}}(\nu)$$

where $B_t^h(\nu)$ is unstable as the labor income $W\iota\gamma^t v_t$ grows over time. Meanwhile the household distribution $\Phi_t^{\text{household}}(\nu)$ is stable. Hence, following the same logic as in, we can divided above equation with γ^t on both side. On LHS the growth rate is normalized by stationary distribution $\Phi^{\text{tra}}(a,e)$ and $\Phi^{\text{mod}}(n,e)$ while on RHS it is normalized by households' standardized saving where $b_t^h(\nu)$ is defined in equation (B.14).

$$\int_{\Lambda\times E} B_t^{\text{tra}}(n,e)d\Phi^{\text{tra}}(n,e) + \int_{\Lambda\times E} B_t^{\text{mod}}(n,e)\Phi^{\text{mod}}(a,e) = \int_V \iota b_t^h(\nu)d\Phi_t^{\text{household}}(\nu) \tag{B.28}$$

When solve the model, following Midrigan and Xu (2014) we normalize all the variables in production sector with the permanent productivity z. Then equations (B.27) and (B.28) become

$$\int_Z\int_{\Lambda\times E} exp(z)l_{n,e}^{\text{tra}}\,d\Phi_{n,e}^{\text{tra}}\,dG_z + \int_Z\int_{\Lambda\times E} exp(z)l_{n,e}^{\text{mod}}\,d\Phi_{n,e}^{\text{mod}}\,dG_z = \iota \tag{B.29}$$

and

$$\int_Z\int_{\Lambda\times E} exp(z)b_{n,e}^{\text{tra}}\,d\Phi_{n,e}^{\text{tra}}\,dG_z + \int_Z\int_{\Lambda\times E} exp(z)b_{n,e}^{\text{mod}}\,d\Phi_{n,e}^{\text{mod}}\,dG_z = \int_V d_t(\nu)d\Phi_t^{\text{household}}(\nu) \tag{B.30}$$

B.4.5 Economy without used capital

Since only the modern sector will use the used capital, when the option to used capital disappears, only the entrepreneurs in modern sector will response to the change. Hence we only consider the optimization problems of the entrepreneurs in modern sector and the agents in other sectors will hold their original policy functions.

When the corporations in modern sector cannot access to the used capital, their optimization problem becomes

$$\pi_t^m(e) = \max_{K_{t-1}^n, L_t} \exp(e_t + \kappa + z)^{1-\eta} \left(L_t^\alpha K_{t-1}^{1-\alpha}\right)^\eta - (r + \delta^n) K_{t-1}^n - W_t L_t$$

$$s.t.(1 + r)\left(K_{t-1}^n - N_{t-1}\right) \leq \theta (1 - \delta^n) K_{t-1}^n$$

and the budget constraint becomes

$$C_t^{\text{mod}} + N_t = \pi_t^m(e) + (1 + r) N_{t-1}$$

where $N_{t-1} = K_{t-1}^n - B_{t-1}^{\text{mod}}$

with the value function

$$V^{\text{mod}}(N_{t-1}, e_t) = \log\left(C_t^{mod}\right) + E_t \beta V^{\text{mod}}(N_t, e_{t+1})$$

$$C_t^{\text{mod}} + N_t = \pi_t^m(e_t) + (1 + r) N_{t-1}$$

For simplicity we further define

$$A_t = \exp(e_t + \kappa)$$

After solving above equations we can get the FOCs of the optimization problems

For non-binding firms

$$K_{t-1} = A_t \left(\frac{r + \delta^n}{(1 - \alpha)\eta}\right)^{\frac{1}{\eta-1}} \left(\frac{W}{r + \delta^n} \frac{1 - \alpha}{\alpha}\right)^{\frac{\alpha\eta}{\eta-1}}$$

$$L_t = A_t \left(\frac{r + \delta^n}{(1 - \alpha)\eta}\right)^{\frac{1}{\eta-1}} \left(\frac{W}{r + \delta^n} \frac{1 - \alpha}{\alpha}\right)^{\frac{\alpha\eta}{\eta-1} - 1}$$

For binding firms

$$K_{t-1} = \frac{1 + r}{1 + r - \theta(1 - \delta^n)} N_{t-1}$$

$$L_t = \left(\frac{\alpha\eta}{W} A^{1-\eta}\right)^{\frac{1}{1-\alpha\eta}} K^{\frac{(1-\alpha)\eta}{1-\alpha\eta}}$$

The threshold of binding condition will be

$$\overline{N}_{t-1} = A_t \left(\frac{r+\delta^n}{(1-\alpha)\eta}\right)^{\frac{1}{\eta-1}} \left(\frac{W}{r+\delta^n}\frac{1-\alpha}{\alpha}\right)^{\frac{\alpha\eta}{\eta-1}} \frac{1+r-\theta(1-\delta^n)}{1+r}$$

C Data Construction

Our firm-level data comes from Compustat-CRSP merged annual data base. The sample period is from 1971 to 2023. We only include manufacturing firm with SIC code between 2000 and 3999. We further drop observations with missing or negative total asset (AT) and sales (SALE). After applying those filters, we end up with a sample with 115,322 firm-year observations. As in Eisfeldt and Rampini (2006), we measure aggregate capital reallocation in each year as the sum of firm-level sales of property (SPPE) and acquisition (AQC), and compute the ratio of aggregate capital reallocation over total firm-level capital expenditure (CAPEX) in each year. We then use the time-series average of this ratio as our measure of the share of investment expenditure on used capital. We follow Ai, Croce, and Li (2013) to measure firm-level output as the difference between sales (SALE) and cost-of-goods-sold (COGS). To compute the investment and debt to output ratio, we first calculate the sum of firm-level capital expenditure (CAPEX), and the sum of firm-level debt (DLC+DLTT) in each year. Then, we get the time-series average of the ratio of total investment and debt over total output. Additionally, we use log of number of employees (EMP) and physical capital (PPENT) as the measure of firm-level employment and total capital, respectively. We compute the firm-level growth rate of output, employment, and capital as the one-year difference of their logged values. For the computation of standard deviation, auto correlation, we first extract the firm-level idiosyncratic components of each measure by obtaining the residuals of regressing the original variables on 4-digit SIC industry-by-year fixed effects. All the variables are winsorized with cutoffs 2% and 98% in each year before the regressions. We then calculate the standard deviation of the residuals using all available observations, and calculate the auto-correlation by regressing the residuals of each variable on its lagged values.

The aggregate level one-year interest rate data ("REAINTRATREARAT1YE") and CPI index ("CAPIAUCSL") come from the website of Federal reserve bank of St. Louis. The interest rate data series is from 1982 to 2023. We use the CPI data to deflate all the variables whenever we need to calculate the time-series average.